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# A Numerical Method for the Computation of Electromagnetic Modes in Optical Fibers

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**Abstract:**

In this article, we propose a new numerical method for the computation of electromagnetic modes in an optical fiber. The main difficulty lies in the fact that we have to solve an eigenvalue problem posed in $\mathbb{R}^2$. We reduce the problem to a disc with the help of the introduction of non local boundary operators that can be expressed in terms of Fourier series. A particular attention is drawn to the respect of the free divergence condition. Finally, our method is reduced to the resolution of a series of fixed point equations related to the eigenvalues of some self-adjoint operators with compact resolvent.

**Key-words:** Electromagnetic Open Waveguides, Maxwell's Equations, Spectral Theory, Transparent Boundary Conditions, Non Local Boundary Operators, Reduction to a Bounded Domain, Selfadjoint Operators

*(Résumé : tsvp)*

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# Méthode Numérique pour le Calcul des Modes Electromagnétiques dans des Fibres Optiques

**Résumé :** Nous proposons dans cet article une nouvelle méthode numérique pour le calcul des modes électromagnétiques guidés par des fibres optiques. La principale difficulté tient au fait que l'on a à résoudre un problème aux valeurs propres dans $\mathbb{R}^2$. Nous réduisons le problème à un disque, moyennant l'introduction d'opérateurs de bord non-locaux s'exprimant à l'aide de séries de Fourier. Une attention particulière est apportée au respect de la condition de divergence nulle. Finalement, notre méthode se ramène à la résolution d'une série d'équations de points fixes relatives aux valeurs propres d'opérateurs auto-adjoints à résolvante compacte.

**Mots-clé :** Guides d'Ondes Ouverts, Electromagnétisme, Equations de Maxwell, Théorie Spectrale, Conditions aux Limites Transparentes, Opérateurs de Bord non Locaux, Réduction à un Domaine Borné, Opérateurs Auto-adjoints

# 1 Mathematical formulation of the problem

## 1.1 Position of the problem

We consider a 3D dielectric linear isotropic medium occupying the whole space $\mathbb{R}^3$. We denote by $(x, x_3)$, with $x = (x_1, x_2) \in \mathbb{R}^2$, the generic point of $\mathbb{R}^3$. We assume that the propagation medium has a cylindrical structure in the sense that it is invariant under any translation in the $x_3$ direction (see figure 1.1). This means that the dielectric permittivity $\varepsilon$ and the magnetic permeability $\mu$ are functions of the only transverse variable $x$. We make the usual assumption on the functions $\varepsilon$ and $\mu$: they are measurable, strictly positive and bounded functions. We introduce

$$\left\{ \begin{array}{l} \varepsilon_- = \inf_{x\in\mathbb{R}^2} \varepsilon(x) > 0 \quad ; \quad \varepsilon^+ = \sup_{x\in\mathbb{R}^2} \varepsilon(x) < +\infty \\ \\ \mu_- = \inf_{x\in\mathbb{R}^2} \mu(x) > 0 \quad ; \quad \mu^+ = \sup_{x\in\mathbb{R}^2} \mu(x) < +\infty \; . \end{array} \right. \tag{1.1}$$

Another important property of the propagation medium we shall consider is the fact that each cross section (i.e. parallel to $(x_1, x_2)$) is homogeneous at infinity. More precisely $\varepsilon$ and $\mu$ are constant outside some bounded domain $B_R$ (the disc of radius $R$ centered at the origin) of the plane $(x_1, x_2)$:

$$\exists R > 0 \quad / \quad |x| \geq R \quad \Rightarrow \quad \varepsilon(x) \;=\; \varepsilon_\infty \quad , \quad \mu(x) \;=\; \mu_\infty \; . \tag{1.2}$$

For the sequel it is useful to introduce the local propagation velocity $c(x)$ of the medium, defined by $c(x)^2 = (\varepsilon(x)\mu(x))^{-1}$ ; $c(x)$ is clearly a bounded, strictly positive measurable function and we shall set $c_- = \inf_{x\in\mathbb{R}^2} c(x), \quad c_+ = \sup_{x\in\mathbb{R}^2} c(x), \quad c_\infty = (\varepsilon_\infty\mu_\infty)^{-\frac{1}{2}}$.

The electromagnetic field is as usual described by the electric field $\mathbb{E}(x, x_3, t)$ and the magnetic field $\mathbb{H}(x, x_3, t)$ ($t > 0$ denotes the time) whose variations are governed by Maxwell's equations

$$\varepsilon \frac{\partial \mathbb{E}}{\partial t} - \mathrm{rot}\mathbb{H} \;=\; 0 \qquad \mu \frac{\partial \mathbb{H}}{\partial t} + \mathrm{rot}\mathbb{E} \;=\; 0. \tag{1.3}$$

**Definition 1.1** *Guided waves are particular solutions of (1.3) on the form*

$$\left\{ \begin{array}{l} \mathbb{E}(x, x_3, t) = (E_1(x) \;,\; E_2(x) \;,\; -iE_3(x))^T \exp i(\omega t - \beta x_3) \\ \\ \mathbb{H}(x, x_3, t) = (H_1(x) \;,\; H_2(x) \;,\; iH_3(x))^T \exp i(\omega t - \beta x_3) \end{array} \right. \tag{1.4}$$

*where*

- $\omega > 0$ *is the pulsation of the wave*
- $\beta > 0$ *is the wavenumber in the* $x_3$*-direction*

*and where the transverse electromagnetic energy is supposed to be finite (we set* $E = (E_1, E_2, E_3)^t$ *and* $H = (H_1, H_2, H_3)^t$*)*

$$\int_{\mathbb{R}^2} (\varepsilon \mid E \mid^2 + \mu \mid H \mid^2) \; dx < +\infty \; . \tag{1.5}$$

The expression (1.4) represents an harmonic plane wave propagating without any distortion in the $x_3$ direction with a velocity $V = \omega/\beta$ (the phase velocity). Such a solution is periodic in the $x_3$ direction and the period $\lambda = 2\pi/\beta$ is called the wavelength. The 2D vector fields (with values in $\mathbb{R}^3$) $E(x)$ and $H(x)$ describe the distribution of the electromagnetic field in each cross section.

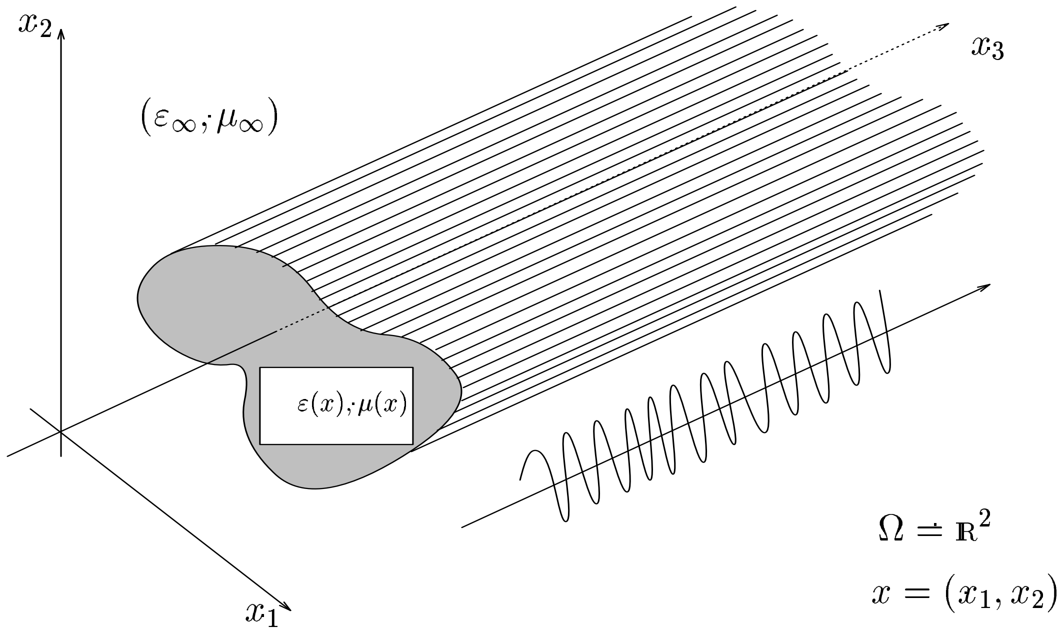



Figure 1.1: Open Waveguide

Guided waves differ from usual plane waves in a homogeneous medium, for instance, by the square integrability condition (1.5) which characterizes the fact that a mode is guided or not. This condition physically means that the energy of the mode remains confined in some bounded region of the cross section: this is where the fact that the coefficients $\varepsilon(x)$ and $\mu(x)$ vary locally plays a fundamental role. Indeed when these coefficients are constant, it is well known that guided waves do not exist.

Before entering the rigorous mathematical treatment, we first need to derive the equations of our problem. Plugging formula (1.4) into (1.3) leads to the following system of equations:

$$\mathrm{rot}^*_\beta H \;=\; \varepsilon\omega E \qquad \mathrm{rot}_\beta E \;=\; \mu\omega H \tag{1.6}$$

where the differential operator $\mathrm{rot}_\beta$ is defined for $u \in \mathcal{D}'(\mathbb{R}^2)$ by

$$\mathrm{rot}_\beta u = \begin{pmatrix} \dfrac{\partial u_3}{\partial x_2} - \beta u_2 \\ \beta u_1 - \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} \end{pmatrix} \tag{1.7}$$

and $\mathrm{rot}^*_\beta$ is the adjoint of $\mathrm{rot}_\beta$, also given by $\mathrm{rot}^*_\beta = \mathrm{rot}_{-\beta}$.

**Remark 1.1** *The change of the functions $E_3$ in $-iE_3$ and $H_3$ in $iH_3$ allows us to work with fields $(E, H)$ in $\mathbb{R}^3$ instead of $\mathbb{C}^3$.*

One can then give a formulation in terms of a symmetric eigenvalue problem, by eliminating for instance $H$ in (1.6), whose unknown is therefore the distribution electric field $E$ in the cross section:

$$\varepsilon^{-1}\mathrm{rot}^*_\beta(\mu^{-1}\mathrm{rot}_\beta E) \;=\; \omega^2 E. \tag{1.8}$$

To complete our presentation, we need a functional framework, which we give in detail in the next section.

## 1.2 Mathematical tools

*We simply present in this section various mathematical tools and related results. Such of them are classical (see for example R. Dautray and J.L. Lions [9], or V. Girault and P.A. Raviart [13]), while other ones have been proven in [16], [26]. We have chosen to include them here for the sake of completeness and to make precise the notation.*

### 1.2.1 Differential Operators

We denote by $\Omega$ an open set of $\mathbb{R}^2$. In the sequel, we keep the notation in bold character for the 2D vector fields ($\boldsymbol{u} = (u_1, u_2)$). We are led to use the following operators.

The scalar differential operators acting on a 2D vector field: Let be $\boldsymbol{u} \in \mathcal{D}'(\Omega)^2$

$$\mathrm{div}\boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \qquad \mathrm{rot}\boldsymbol{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

The vectorial 2D differential operators acting on a scalar function: Let be $\varphi \in \mathcal{D}'(\Omega)$

$$\nabla\varphi = (\frac{\partial \varphi}{\partial x_1}\ ,\ \frac{\partial \varphi}{\partial x_2}) \qquad \vec{rot}\varphi = (\frac{\partial \varphi}{\partial x_2}\ ,\ -\frac{\partial \varphi}{\partial x_1})$$

The scalar differential operators acting on a scalar field: Let be $\varphi \in \mathcal{D}'(\Omega)$

$$\triangle\varphi \;=\; \mathrm{div}(\nabla\varphi) \;=\; \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2}$$

The differential operators depending on the parameter $\beta$:

For $u = (\boldsymbol{u}, u_3) \in \mathcal{D}'(\Omega)^2 \times \mathcal{D}'(\Omega)$ $\quad \mathrm{div}_\beta u = \mathrm{div}\boldsymbol{u} - \beta u_3$

For $\varphi \in \mathcal{D}'(\Omega)$ $\quad \nabla_\beta\varphi = (\nabla\varphi\ ,\ \beta\varphi)$

$$\triangle_\beta\varphi \;=\; \mathrm{div}_\beta(\nabla_\beta\varphi) \;=\; \triangle\varphi - \beta^2\varphi$$

From the well known relations $\mathrm{rot}(\nabla\,.\,) = 0$ and $\mathrm{div}(\vec{rot}\,.\,) = 0$, we deduce the following ones:

$$\forall\varphi \in \mathcal{D}'(\Omega) \quad \mathrm{rot}_\beta(\nabla_\beta\varphi) = 0 \tag{1.9}$$

$$\forall u \in \mathcal{D}'(\Omega)^3 \quad \mathrm{div}_\beta(\mathrm{rot}^\star_\beta u) = 0. \tag{1.10}$$

### 1.2.2 Functional spaces, trace theorems and Green's formulae

We will denote by $(u, v)$ the inner product of $u$ and $v$ on $L^2(\Omega)^d$ where $d$ will be equal to 1,2 or 3. The norm associated to this inner product will be denoted by $\|\quad\|$. In a general way, the usual norm of the Sobolev space $H^s(\Omega)$ for $s \in \mathbb{R}$, will be denoted by $\|\quad\|_s$ and the associated scalar product by $(\quad,\quad)_s$.

We introduce the Hilbert space

$$H(\mathrm{rot}_\beta, \Omega) = \{\ u \in L^2(\Omega)^3\ ;\ \mathrm{rot}_\beta u \in L^2(\Omega)^3\ \}$$

that we equip with the scalar product $(.,.)_{H(\mathrm{rot}_\beta,\Omega)}$ defined by

$$(u, v)_{H(\mathrm{rot}_\beta,\Omega)} = (u, v) + (\mathrm{rot}_\beta u, \mathrm{rot}_\beta v).$$

$H(\mathrm{rot}_\beta,\Omega)$ is isomorphic to $H(\mathrm{rot},\Omega)\times H^1(\Omega)$ and the norm $\|.\|_{H(\mathrm{rot}_\beta,\Omega)}$ is equivalent to the norm $\|.\|_{H(\mathrm{rot},\Omega)\times H^1(\Omega)}$. One also has

$$H_0(\mathrm{rot}_\beta,\Omega) := \overline{\mathcal{D}(\Omega)^3}^{H(\mathrm{rot}_\beta,\Omega)} = H_0(rot,\Omega)\times H_0^1(\Omega)$$

where

$$\begin{cases} H_0(rot,\Omega) = \{\boldsymbol{u}\in H(\mathrm{rot},\Omega) \quad \boldsymbol{u}\wedge n = 0\} \\ H_0^1(\Omega) \ = \ \{\ u\in H^1(\Omega) \quad u_{/\partial\Omega} = 0\ \}. \end{cases}$$

**Definition 1.2** *Let us set*

$$H^{-\frac12}(rot_\beta,\Gamma) \ = \ H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma) \quad , \quad H^{-\frac12}(div_\beta,\Gamma) \ = \ H^{\frac12}(\Gamma)\times H^{-\frac12}(\Gamma).$$

**Remark 1.2** *One can also define these two spaces as:*

$$\begin{cases} H^{-\frac12}(rot_\beta,\Gamma) \ = \ \{\ u\in H^{-\frac12}(\Gamma)^2\ ;\ rot_{\beta,\Gamma}u\in H^{-\frac12}(\Gamma)\ \} \\ H^{-\frac12}(div_\beta,\Gamma) \ = \ \{\ u\in H^{-\frac12}(\Gamma)^2\ ;\ div_{\beta,\Gamma}u\in H^{-\frac12}(\Gamma)\ \}. \end{cases}$$

*for convenient tangential $rot_{\beta,\Gamma}$ and $div_{\beta,\Gamma}$ operators (see appendix B of [26]).*

It is well known that, if $\Omega$ is an open set of $\mathbb{R}^2$, Lipschitz continuous, the trace operator $\gamma_t$ (resp. $\gamma_\tau$), such that

$$\begin{cases} \gamma_t(u) \ := \ u\wedge n \ := (u_{3/\Gamma},\boldsymbol{u}\wedge n_{/\Gamma}) \\ \gamma_\tau(u) \ := \ (u\wedge n)\wedge n \ := \ (\boldsymbol{u}\wedge n_{/\Gamma},-u_{3/\Gamma}). \end{cases} \quad \text{where } u=(\boldsymbol{u},u_3)\in H(\mathrm{rot}_\beta,\Omega) \tag{1.11}$$

is continuous from $H(\mathrm{rot}_\beta,\Omega)$ into $H^{-\frac12}(\mathrm{div}_\beta,\Gamma)$ (resp. $H^{-\frac12}(\mathrm{rot}_\beta,\Gamma)$) ([13]). Moreover,

**Proposition 1.1**

$$Ker(\gamma_t) = Ker(\gamma_\tau) = H_0(rot_\beta,\Omega) \tag{1.12}$$

*and $\gamma_t$ (resp. $\gamma_\tau$) is continuous and surjective onto $H^{-\frac12}(div_\beta,\Gamma)$ (resp. $H^{-\frac12}(rot_\beta,\Gamma)$).*

From these definitions, we can generalize some Green's formulae (see V. Girault and P.A. Raviart [13]) to $H(\mathrm{rot}_\beta,\Omega)$.

**Theorem 1.1** *Green 's Formula in $H(rot_\beta,\Omega)$:* $\quad \forall(u,v)\in H(rot_\beta,\Omega)^2$

$$(rot_\beta u,v) = (u,rot_\beta^\star v) + < u\wedge n,(v\wedge n)\wedge n > \tag{1.13}$$

$$(rot_\beta u,v) = (u,rot_\beta^\star v) - < v\wedge n,(u\wedge n)\wedge n > \tag{1.14}$$

*where $<.,.>$ represents the duality between $H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)$ and $H^{\frac12}(\Gamma)\times H^{-\frac12}(\Gamma)$.*

In the same way, we define the Hilbert space

$$H(\mathrm{div}_\beta,\Omega) = \{\ u\in L^2(\Omega)^3\ ;\ \mathrm{div}_\beta u\in L^2(\Omega)^3\ \}$$

equipped with the inner product $(.,.)_{H(\mathrm{div}_\beta,\Omega)}$ defined by

$$(u,v)_{H(\mathrm{div}_\beta,\Omega)} = (u,v) + (\mathrm{div}_\beta u,\mathrm{div}_\beta v).$$

This space is isomorphic to $H(\mathrm{div},\Omega)\times L^2(\Omega)$ and we have the following Green's formula

$$\forall(u,\phi)\in H(\mathrm{div}_\beta,\Omega)\times H^1(\Omega) \quad (\mathrm{div}_\beta u,\phi) \ + \ (u,\nabla_\beta\phi) = < \boldsymbol{u}\cdot\boldsymbol{n},\phi > \tag{1.15}$$

where $<\ .\ ,\ .\ >$ indicates the duality product between $H^{-\frac12}(\Gamma)$ and $H^{\frac12}(\Gamma)$.

### 1.2.3 Decomposition of the space $L^2(\Omega)^3$

We introduce the Hilbert space $H_\varepsilon(\Omega) \;:=\; L^2(\Omega)^3$ equipped with the weighted inner product

$$(u,v)_\varepsilon = (\varepsilon u, v).$$

We are going to give here an orthogonal decomposition of $H_\varepsilon(\Omega)$ whose interest will appear in the next section. Let us enounce the

**Theorem 1.2** *$H_\varepsilon(\Omega)$ admits the orthogonal decomposition (for the scalar product $(\ ,\ )_\varepsilon$ )*

$$H_\varepsilon(\Omega) \;=\; H_{\varepsilon,\beta}(\Omega) \;\oplus\; H^{\perp}_{\varepsilon,\beta}(\Omega) \qquad \textit{with } \beta \neq\; 0 \tag{1.16}$$

$$\textit{where} \quad H_{\varepsilon,\beta}(\Omega) := \{\; u \in H_\varepsilon(\Omega)\;;\; div_\beta(\varepsilon u) = 0 \;\}$$

$$\textit{and} \qquad H^{\perp}_{\varepsilon,\beta}(\Omega) = \{\; \nabla_\beta \varphi\;;\; \varphi \in H^1_0(\Omega) \;\}$$

**Proof** For the proof, we refer to [26].

♦

**Remark 1.3** *Thanks to the previous lemmas, we obtain a characterization of $H_{\varepsilon,\beta}(\Omega)$ and of $H^{\perp}_{\varepsilon,\beta}(\Omega)$:*

$$\left\{\begin{array}{l} H_{\varepsilon,\beta}(\Omega) = \{\; u = (\boldsymbol{u}, u_3) \in H_\varepsilon\;;\; u \in H(div_\varepsilon, \Omega) \textit{ and } u_3 = \dfrac{1}{\varepsilon\beta} div(\varepsilon \boldsymbol{u}) \;\} \\ H^{\perp}_{\varepsilon,\beta}(\Omega) = \{\; u = (\boldsymbol{u}, u_3)\;;\; u_3 \in H^1_0(\Omega) \textit{ and } \boldsymbol{u} = \dfrac{1}{\beta}\nabla u_3 \;\} \end{array}\right.$$

$$\textit{where} \quad H(div_\varepsilon, \Omega) = \{\; \boldsymbol{u} \in L^2(\Omega)^2\;;\; div(\varepsilon \boldsymbol{u}) \in L^2(\Omega) \;\}.$$

*In other words, $H_{\varepsilon,\beta}(\Omega)$ is isomorphic to a space of 2D vector fields $H(div_\varepsilon, \Omega)$ , whereas $H^{\perp}_{\varepsilon,\beta}(\Omega)$ is isomorphic to $H^1_0(\Omega)$.*

We have also the following characterization of $H_{\varepsilon,\beta}(\Omega)$.

**Theorem 1.3** *Let us consider now $\Omega$ as a bounded open set Lipschitz continuous, then*

$$H_{\varepsilon,\beta}(\Omega) \;=\; \{\; \frac{1}{\varepsilon} rot^{\star}_\beta v\;;\; v \in H(rot_\beta, \Omega) \;\}. \tag{1.17}$$

**Proof** The proof, which is given in detail in [26], follows the one given in [9], for an analogous result in dimension 3.

♦

## 1.3 Mathematical formulation of the problem

We denote by $\tilde{A}_{\varepsilon,\beta}$ the unbounded operator in $H_\varepsilon$ defined by

$$\left\{\begin{array}{l} D(\tilde{A}_{\varepsilon,\beta}) = \{u \in H(\mathrm{rot}_\beta, \mathbb{R}^2)\; /\; \mathrm{rot}^*_\beta(\mu^{-1}\mathrm{rot}_\beta u) \in H_\varepsilon(\mathbb{R}^2)\} \\ \tilde{A}_{\varepsilon,\beta} u \;=\; \varepsilon^{-1}\mathrm{rot}^*_\beta(\mu^{-1}\mathrm{rot}_\beta u), \quad \forall u \in D(\tilde{A}_{\varepsilon,\beta})\;. \end{array}\right. \tag{1.18}$$

Therefore the problem to be solved can be written as, for a given value of the wavenumber $\beta$ considered as a parameter

$$\left\{\begin{array}{l} \text{Find } E \in D(\tilde{A}_{\varepsilon,\beta}) \text{ and } \omega^2 \in \mathbb{R}^{+*} \quad \text{such that} \\ \tilde{A}_{\varepsilon,\beta} E \;=\; \omega^2 E, \quad E \neq 0. \end{array}\right. \tag{1.19}$$

The problem (1.19) clearly appears as a family of eigenvalue problems parameterized by $\beta$ in which $\omega^2$ plays the role of the eigenvalue and $E$ the role of the corresponding eigenvector. Two properties have to be emphasized.

(i) From (1.10), we see that, as soon as $\omega^2 \neq 0$,

$$\mathrm{div}_\beta(\varepsilon E) = 0 \tag{1.20}$$

which means that all physically relevant solutions (i.e. for which $\omega^2 \neq 0$) satisfy the generalized free divergence condition (1.20).

(ii) We deduce from (1.9) that for any $\varphi$ in $H^1(\mathbb{R}^2)$, $\nabla_\beta\varphi$ belongs to $D(\tilde{A}_{\varepsilon,\beta})$ and $\tilde{A}_{\varepsilon,\beta}\nabla_\beta\varphi = 0$.

We exploit these properties in the following lemma, where the orthogonal decomposition (1.16) of $H_\varepsilon(\mathbb{R}^2)$ will take its entire interest.

**Lemma 1.1**

$$Ker\tilde{A}_{\varepsilon,\beta} = \{\nabla_\beta\varphi,\ \varphi \in H^1(\mathbb{R}^2)\} \ \ and\ \ \mathcal{I}m\tilde{A}_{\varepsilon,\beta} \subset H_{\varepsilon,\beta}(\mathbb{R}^2)$$

**Proof** See proof of Lemma (1.2) of [16].

♦

Combining condition (1.20) and Lemma 1.1, it is natural to consider the restriction of the operator $\tilde{A}_{\varepsilon,\beta}$ to the space $H_{\varepsilon,\beta}(\mathbb{R}^2)$, which is a closed subspace of $H_\varepsilon(\mathbb{R}^2)$ (and then an Hilbert space for the inner product $(.,.)_\varepsilon$). We shall consider this restriction as an unbounded operator $A_{\varepsilon,\beta}$ in the Hilbert space $H_{\varepsilon,\beta}(\mathbb{R}^2)$

$$\left\{\begin{array}{lcl} D(A_{\varepsilon,\beta}) & = & \{u \in H(\mathrm{rot}_\beta, \mathbb{R}^2) \cap H_{\varepsilon,\beta}(\mathbb{R}^2) \ / \ \mathrm{rot}^*_\beta(\mu^{-1}\mathrm{rot}_\beta u) \in L^2(\mathbb{R}^2)^3\} \\ A_{\varepsilon,\beta}u & = & \varepsilon^{-1}\mathrm{rot}^*_\beta(\mu^{-1}\mathrm{rot}_\beta u)\ . \end{array}\right. \tag{1.21}$$

Finally, because of (1.20), problem (1.19) can be reduced to

$$\left\{\begin{array}{l} \text{Find } E \in D(A_{\varepsilon,\beta}), \omega^2 > 0 \quad / \\ A_{\varepsilon,\beta}E \ = \ \omega^2 E \quad , \quad E \neq 0. \end{array}\right. \tag{1.22}$$

For each $\beta$, we have to determine the point spectrum of $A_{\varepsilon,\beta}$. Because of the unboundedness of $\mathbb{R}^2$, the embedding of $D(A_{\varepsilon,\beta})$ in $H_\varepsilon$ is not compact: this is why the existence of this point spectrum is not a trivial question. Formulation (1.25) is the one we shall use for our analysis. It differs essentially from the one of A. Bamberger and A.S. Bonnet-Ben Dhia [2] in the fact that the generalized free divergence condition (1.20) has been incorporated in the functional space $H_{\varepsilon,\beta}$. This is essential in order to get some local compactness.

In the sequel we shall need to work with the bilinear form $a_\varepsilon(\beta\ ;.,.)$ associated with $A_{\varepsilon,\beta}$. This bilinear form is defined on the space

$$\begin{array}{lcl} V_{\varepsilon,\beta}(\mathbb{R}^2) & = & H(\mathrm{rot}_\beta, \mathbb{R}^2) \cap H_{\varepsilon,\beta}(\mathbb{R}^2) \\ & = & \{u = (\boldsymbol{u}, u_3) \in H(\mathrm{rot}, \mathbb{R}^2) \times H^1(\mathbb{R}^2) \ / \ \mathrm{div}_\beta(\varepsilon u) = 0\} \end{array} \tag{1.23}$$

and its expression is given by

$$a_\varepsilon(\beta\ ; u, v) = \int_{\mathbb{R}^2} \mu^{-1}\mathrm{rot}_\beta u \cdot \mathrm{rot}_\beta v \ dx \quad \forall (u,v) \in V_{\varepsilon,\beta}(\mathbb{R}^2). \tag{1.24}$$

By Green's formula, one has

$$(A_{\varepsilon,\beta}u,v)_\varepsilon \;=\; a_\varepsilon(\beta\,;u,v) \quad \forall (u,v)\in D(A_{\varepsilon,\beta})\times V_{\varepsilon,\beta}. \tag{1.25}$$

The fact that $a_\varepsilon(\beta\,;.,.)$ is symmetric and positive implies that $A_{\varepsilon,\beta}$ is symmetric and positive. Moreover we can prove that it is self-adjoint and bounded from below. We recall here some results concerning the spectral analysis, which can be found in [16]. In what follows, $\sigma(A_{\varepsilon,\beta})$ denotes the spectrum of $A_{\varepsilon,\beta}$ and $\sigma_{\text{ess}}(A_{\varepsilon,\beta})$ its essential spectrum.

(i) $\sigma(A_{\varepsilon,\beta}) \subset [c_-^2\beta^2,+\infty[$ and $\sigma_{\text{ess}}(A_{\varepsilon,\beta}) = [c_\infty^2\beta^2,+\infty[$.

(ii) There are no eigenvalues embedded in $]c_\infty^2\beta^2,+\infty[$.

(iii) The point spectrum of $A_{\varepsilon,\beta}$ is finite for each $\beta>0$.

(iv) As soon as $c_- < c_\infty$, there exists a sequence $(\beta_m^*)\subset \mathbb{R}$ called upper thresholds such that

$$\begin{aligned}\beta_m^* \;=\; \inf\{\quad \beta_m>0 \quad / \quad \forall\beta>\beta_m,\ A_{\varepsilon,\beta} \text{ admits at least } m \\ \text{eigenvalues strictly smaller than } c_\infty^2\beta^2\ \}.\end{aligned} \tag{1.26}$$

The last result requires some weak regularity assumptions about $\varepsilon$ and $\mu$ that can be found in [16], as well as more detail results about the point spectrum of $A_{\varepsilon,\beta}$.

From now on, we assume that we are in the case $c_- < c_\infty$ and our goal is to design a method for the numerical computation of the eigenvalues $\omega^2$ of $A_{\varepsilon,\beta}$. Of course because of properties of (i),(ii), these eigenvalues satisfy

$$c_-^2\beta^2 \;<\; \omega^2 \;<\; c_\infty^2\beta^2. \tag{1.27}$$

From now on, we shall consider values of $(\beta,\omega)$ satisfying these double inequalities. More exactly, we are going to construct a new problem posed in a bounded domain, of which we will analyze the mathematical properties and prove the equivalence, in a sense we shall make precise, with the initial problem posed in the whole space $\mathbb{R}^3$.

The interest of this new problem lies in the fact that one is reduced to deal with self-adjoint operators with compact resolvent. Such operators have a pure point spectrum, which is particularly convenient from the numerical point of view. It is known that contrary to the continuous spectrum, the discrete spectrum is stable with respect to a discretization, provided one treats correctly the problem of spurious modes (see A. Bossavit [6] [5], P. Joly and al. [17]). However, the price of this simplification appears though two additional difficulties:

(i) the introduction of two non local operators acting on circular artificial boundary, which allow us to write the transparent boundary conditions.

(ii) the introduction of an artificial non linearity. Indeed, the new operators depend on the eigenvalues we are looking for: we thus have to treat a fixed point problem which corresponds to an artificial non linearity.

The framework of this article is the following one. We shall obtain a first reduction of the problem in a bounded domain, namely the ball of radius $R$, with the help of the introduction of a first boundary operator $T_R(\omega,\beta)$. This first step is not satisfactory, in particular because we do not know how analyze the properties of the new operator: (in particular, one does not know if its spectrum is purely punctual). The key for removing this difficulty consists in introducing a new trace operator $S_R(\omega,\beta)$, which allows us to take into account the generalized free divergence condition and to get some desired results of compactness (see section 3).

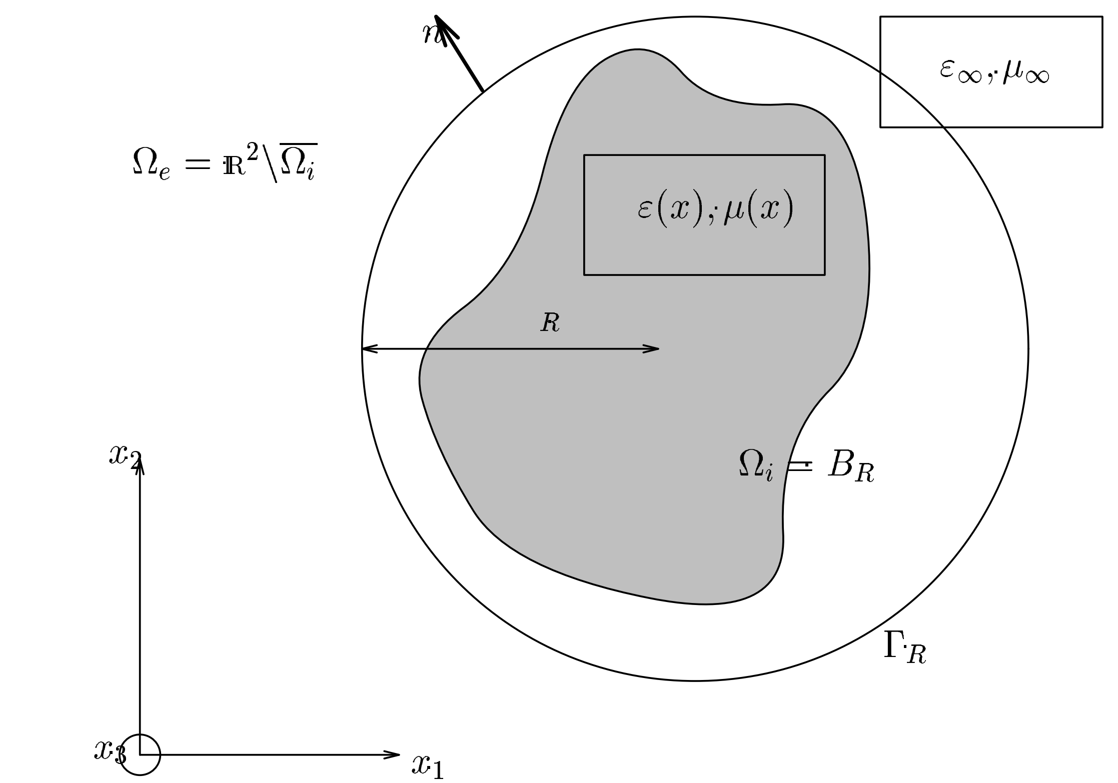



Figure 1.2: Description of $\Omega_i$ and $\Omega_e$

This second step allows us to define a new problem, always posed in a bounded domain, that we shall prove to be equivalent to the initial one (cf. Theorem 4.2) and whose main properties will be given. Finally, we prove that the study of guided modes arises from the solving of a countable infinity non linear equations in the plane $(\omega, \beta)$, involving eigenvalues of a new self-adjoint operator $A_{\omega,\beta}$, having compact resolvent (see section 4).
We draw attention of the reader to the most important three results of this section, which are:

- the result of compactness, stated in Theorem 3.2, which allows us to define carefully the operators $A_{\omega,\beta}$,
- the generalized Helmholtz decomposition (Theorem 4.1), which interfere on an essential manner in the proof of the equivalence between the initial eigenvalue problem and the new one,
- the coercivity result established in Theorem 4.3, which is the key for the study of the properties of the operators $A_{\omega,\beta}$.

## 2 A first reduction

*This paragraph is particularly devoted to the introduction of a problem posed in a bounded domain, involving an operator $T_R(\omega, \beta)$ relating the traces of $(u \wedge n) \wedge n$ and of $rot_\beta u \wedge n$ on the artificial boundary $\Gamma_R$ of the ball $B_R$, including the heart of the fiber and we will give its properties. Finally, we emphasize the difficulties of this new problem.*

## 2.1 The transmission problem

We recall that our aim is to solve the following problem: the parameter $\beta$ being strictly positive,

$$(P) \quad \begin{cases} \text{Find} \quad (\omega^2, u) \in \mathbb{R}^{+\star} \times H(\mathrm{rot}_\beta, \mathbb{R}^2) \backslash \{0\} \\ \mathrm{rot}_\beta^\star(\mu^{-1} \mathrm{rot}_\beta u) = \omega^2 \varepsilon u. \end{cases}$$

Let $\Omega$ be the bounded open set of $\mathbb{R}^2$ where the coefficients $\varepsilon$ and $\mu$ vary, namely the heart of the fiber, we choose $R$ large enough such that, if $B_R$ denotes the disc of radius $R$:(see figure 1.2)

$$\Omega \ \subset\subset \ B_R.$$

We then split $\mathbb{R}^2$ into an interior domain $\Omega_i$ and an exterior domain $\Omega_e$ by setting

$$\Omega_i \ = \ B_R \qquad \Omega_e = \mathbb{R}^2 \setminus \overline{B_R} \qquad \Gamma_R \ = \ \partial\Omega_i.$$

Also in the following, we shall set: $u_i = u_{/\Omega_i}$ and $u_e = u_{/\Omega_e}$. Our goal is to limit the effective computations to $\Omega_i$, i.e. to formulate a problem whose only unknown is the field $u_i$. The way we shall limit the domain of computation arises from the classical transmission relations satisfied by solutions $u$ of $(P)$ through any curve and that we recall below in an appropriate form:

**Lemma 2.1** *The eigenpair $(\omega^2, u)$ is solution of $(P)$ if and only if $u \neq 0$ and, $u_i$ and $u_e$ being respectively the restrictions of $u$ on $\Omega_i$ and on $\Omega_e$, the pair $(\omega^2, u_i)$ is solution of:*

$$(P^i) \quad \begin{cases} \text{Find} \quad (\omega^2, u_i) \in \mathbb{R}^{+\star} \times H(rot_\beta, \Omega_i) \\ rot_\beta^\star(\mu^{-1} rot_\beta u_i) = \omega^2 \varepsilon u_i, \end{cases}$$

*the pair $(\omega^2, u_e)$ is solution of:*

$$(P^e) \quad \begin{cases} \text{Find} \quad (\omega^2, u_e) \in \mathbb{R}^{+\star} \times H(rot_\beta, \Omega_e) \\ rot_\beta^\star(\mu^{-1} rot_\beta u_e) = \omega^2 \varepsilon u_e \end{cases}$$

*and $u$ satisfies the following transmission relations:*

$$\left| \begin{array}{rcll} (u_i \wedge n) \wedge n & = & (u_e \wedge n) \wedge n & on \ \Gamma_R \\ rot_\beta u_i \wedge n & = & rot_\beta u_e \wedge n & on \ \Gamma_R \end{array} \right. \tag{2.1}$$

*where $n$ is the unit outward-pointing normal vector field on $\Omega_i$.*

Therefore, if we can construct an operator $T_R(\omega, \beta)$ relating the tangential components of $u_e$ and of $\mathrm{rot}_\beta u_e$ on $\Gamma_R$ as:

$$T_R(\omega, \beta)((u_e \wedge n) \wedge n_{/\Gamma_R}) = \mathrm{rot}_\beta u_e \wedge n_{/\Gamma_R}$$

then $u_i$ will verify the boundary condition on $\Gamma_R$:

$$T_R(\omega, \beta)((u_i \wedge n) \wedge n_{/\Gamma_R}) = \mathrm{rot}_\beta u_i \wedge n_{/\Gamma_R}.$$

We will show now that we can give explicitly the expression of the field outside the heart, with respect to his tangential trace on $\Gamma_R$. Thereafter, we will be able to define rigorously and compute explicitly the operator $T_R(\omega, \beta)$.

## 2.2 The operator $T_R(\omega, \beta)$ and its properties

Let us define a problem denoted $(P^e_{\omega,\beta})$ posed in the homogeneous medium $\Omega_e$:

**Lemma 2.2** *Let be $\omega$ and $\beta$ two parameters strictly positive satisfying the condition*

$$\beta^2 - \frac{\omega^2}{c_\infty^2} > 0.$$

*Then, to each $\varphi_T \in H^{-\frac{1}{2}}(rot_\beta, \Gamma_R)$ corresponds a unique field $u_e \in H(rot_\beta, \Omega_e)$ solution of the problem:*

$$(P^e_{\omega,\beta}) \quad \left\{ \begin{array}{l} Find \quad u_e \in H(rot_\beta, \Omega_e) \\ \\ rot^\star_\beta(rot_\beta u_e) = \dfrac{\omega^2}{c_\infty^2} u_e \\ \\ (u_e \wedge n) \wedge n = \varphi_T \ \ on\ \Gamma_R \end{array} \right. .$$

*Moreover, the map associating to $\varphi_T$ the solution $u_e$ of $(P^e_{\omega,\beta})$ is continuous from $H^{-\frac{1}{2}}(rot_\beta, \Gamma_R)$ into $H(rot_\beta, \Omega_e)$.*

**Proof** 1) - The first step of the proof consists in showing the existence of $\phi \in H(\mathrm{rot}_\beta, \Omega_e)$ such that $(\phi \wedge n) \wedge n_{/\Gamma_R} = \varphi_T$ and $\mathrm{div}_\beta(\phi) = 0$ in $\Omega_e$. First, as the trace mapping $\gamma_\tau$ is continuous and surjective from $H^{-\frac{1}{2}}(\mathrm{rot}_\beta, \Gamma_R)$ into $H(\mathrm{rot}_\beta, \Omega_e)$, there exists $\phi_0 \in H(\mathrm{rot}_\beta, \Omega_e)$ such that $(\phi_0 \wedge n) \wedge n_{/\Gamma} = \varphi_T$, and moreover such that the mapping $\varphi_T \to \phi_0$ is continuous. However $\phi_0$ is not necessarily divergence_$\beta$ free. Nevertheless, from $\phi_0$, we can construct an other field $\phi$, solution of the problem:

$$\left\{ \begin{array}{ll} \text{Find} \quad \phi \in H(\mathrm{rot}_\beta, \Omega_e) \text{ such that} & \\ \\ \mathrm{rot}^\star_\beta(\mathrm{rot}_\beta \phi) + \phi = 0 & \text{in } \Omega_e \\ \\ (\phi \wedge n) \wedge n = \varphi_T & \text{on } \Gamma_R \end{array} \right. \tag{2.2}$$

The problem (2.2) has clearly a unique solution, thanks to Lax-Milgram's theorem. We show easily that the mapping $\varphi_T \to \phi$ remains continuous. Moreover, from (2.2) and the fact that $\mathrm{div}_\beta(\mathrm{rot}^\star_\beta \psi) = 0 \ \forall \psi \in H(\mathrm{rot}_\beta, \Omega_e)$, we show that $\phi$ satisfies the free divergence_$\beta$ condition $(\mathrm{div}_\beta(\phi) = 0)$.
2) - In the second step, we write a variational formulation of the problem $(P^e_{\omega,\beta})$. After a change of variable $v = u - \phi$, we remark that $u$ solution of $(P^e_{\omega,\beta})$ satisfies necessarily $\mathrm{div}_\beta(u) = 0$. The problem $(P^e_{\omega,\beta})$ has then the following variational formulation:

$$\left\{ \begin{array}{l} \text{Find } v \in H_0(rot_\beta, \Omega_e) \text{ such that } \mathrm{div}_\beta(v) = 0 \\ \displaystyle\int_{\Omega_e} \mathrm{rot}_\beta v \cdot \mathrm{rot}_\beta \psi \, dx - \frac{\omega^2}{c_\infty^2} \int_{\Omega_e} v \cdot \psi \, dx = (1 + \frac{\omega^2}{c_\infty^2}) \int_{\Omega_e} \phi \cdot \psi \, dx \\ \forall \psi \in H_0(rot_\beta, \Omega_e). \end{array} \right. \tag{2.3}$$

It is not so obvious that this problem has a solution, contrary to problem (2.2), because of the negative sign in front of the $L^2$ inner product of $v$ and $\psi$.

The orthogonal decomposition (1.16) and the fact that $H_\beta(\Omega_e)^\perp \subset H_0(rot_\beta, \Omega_e)$ lead to the orthogonal decomposition of $H_0(rot_\beta, \Omega_e)$:

$$H_0(rot_\beta, \Omega_e) = V^0_\beta(\Omega_e) \oplus H_\beta(\Omega_e)^\perp \tag{2.4}$$

$$\text{where} \quad \left\{ \begin{array}{lcl} V^0_\beta(\Omega_e) & = & \{u \in H_0(\mathrm{rot}_\beta, \Omega_e)\ ;\ \mathrm{div}_\beta(u) = 0\} \\ H_\beta(\Omega_e)^\perp & = & \{\ \nabla_\beta \phi; \phi \in H^1_0(\Omega_e)\ \} \end{array} \right.$$

and from the fact that $\mathrm{rot}_\beta(\nabla_\beta \phi) = 0$ for each $\phi \in H_0(\mathrm{rot}_\beta, \Omega_e)$, we deduce that $(P^e_{\omega,\beta})$ is equivalent to

$$\left\{ \begin{array}{l} \text{Find} \quad v = u - \phi \in V^0_\beta(\Omega_e) \text{ such that} \\ c_{\omega,\beta}(v, \psi) = (1 + \dfrac{\omega^2}{c^2_\infty}) \displaystyle\int_{\Omega_e} \phi \cdot \psi \, dx \quad \forall \psi \in V^0_\beta(\Omega_e) \end{array} \right. \tag{2.5}$$

where

$$c_{\omega,\beta}(v, \psi) = \int_{\Omega_e} \mathrm{rot}_\beta v \cdot \mathrm{rot}_\beta \psi \, dx - \frac{\omega^2}{c^2_\infty} \int_{\Omega_e} v \cdot \psi \, dx.$$

There remains to show that the bilinear form $c_{\omega,\beta}$ is $V^0_\beta(\Omega_e)$_elliptic, which is exposed in the following lemma. Then, using again the Lax-Milgram's theorem, we can conclude that the exterior problem $(P^e_{\omega,\beta})$ is well posed. The continuity of the map, which associates to each $\phi_T$ the solution $u$ of $(P^e_{\omega,\beta})$ comes from the coercivity of $c_{\omega,\beta}$ and from the continuity of the mapping $\phi_T \to \phi$ (cf. (2.2)).

♦

Let us show the

**Lemma 2.3** *If* $\beta^2 - \dfrac{\omega^2}{c^2_\infty} > 0$*, the bilinear form* $c_{\omega,\beta}$ *is* $V^0_\beta(\Omega_e)$*_elliptic.*

**Proof** For each $v = (\boldsymbol{v}, v_3) \in V^0_\beta(\Omega_e)$, one has

$$\begin{aligned} c_{\omega,\beta}(v, v) &= \int_{\Omega_e} |\mathrm{rot}_\beta v|^2 \, dx - \frac{\omega^2}{c^2_\infty} \int_{\Omega_e} |v|^2 \, dx \\ &= \int_{\Omega_e} |rot\boldsymbol{v}|^2 \, dx + \int_{\Omega_e} |\nabla v_3 - \beta \boldsymbol{v}|^2 \, dx - \frac{\omega^2}{c^2_\infty} \int_{\Omega_e} |v|^2 \, dx \\ &= \int_{\Omega_e} |rot\boldsymbol{v}|^2 \, dx + \int_{\Omega_e} |\nabla v_3|^2 \, dx + \beta^2 \int_{\Omega_e} |\boldsymbol{v}|^2 \, dx \\ &\quad -2\beta \int_{\Omega_e} \nabla v_3 \cdot \boldsymbol{v} \, dx - \frac{\omega^2}{c^2_\infty} \int_{\Omega_e} |v|^2 \, dx. \end{aligned}$$

Now since $v_3 = 0$ on $\Gamma_R$ and $\mathrm{div}\boldsymbol{v} = \beta v_3$ in $\Omega_e$, one has

$$-2\beta \int_{\Omega_e} \nabla v_3 \cdot \boldsymbol{v} \, dx \ = \ 2\beta \int_{\Omega_e} v_3 \mathrm{div}\boldsymbol{v} \, dx \ = \ 2\beta^2 \int_{\Omega_e} v_3^2 \, dx$$

thus

$$\begin{aligned} c_{\omega,\beta}(v, v) &= |rot\boldsymbol{v}|^2_{0,\Omega_e} + (\beta^2 - \frac{\omega^2}{c^2_\infty})|\boldsymbol{v}|^2_{0,\Omega_e} + |\nabla v_3|^2_{0,\Omega_e} + (2\beta^2 - \frac{\omega^2}{c^2_\infty})|v_3|^2_{0,\Omega_e} \\ &\geq \ \inf(1, \beta^2 - \frac{\omega^2}{c^2_\infty})\|\boldsymbol{v}\|^2_{H(rot,\Omega_e)} \ + \ \inf(1, 2\beta^2 - \frac{\omega^2}{c^2_\infty})\|v_3\|^2_{1,\Omega_e} \end{aligned}$$

which concludes the proof, if $\beta^2 - \dfrac{\omega^2}{c^2_\infty} > 0$.

♦

As the solution $u$ of $(P^e_{\omega,\beta})$ satisfies $\mathrm{rot}^\star_\beta(\mathrm{rot}_\beta u) \in L^2(\Omega_e)$, the tangential trace of $\mathrm{rot}_\beta u$ on $\Gamma_R$, that is $\mathrm{rot}_\beta u \wedge n$, is defined in $H^{-\frac{1}{2}}(\mathrm{div}_\beta, \Gamma_R)$. We can then define the trace operator $T_R(\omega, \beta)$.

**Definition 2.1** *Let $\omega$ and $\beta$ be two strictly positive real numbers satisfying the inequality*

$$\beta^2 - \frac{\omega^2}{c_\infty^2} \ > \ 0.$$

*We denote by $T_R(\omega,\beta)$ the linear and continuous mapping from $H^{-\frac{1}{2}}(rot_\beta,\Gamma_R)$ to $H^{-\frac{1}{2}}(div_\beta,\Gamma_R)$ which associates to each $\phi_T$ the tangential trace $rot_\beta u_e \wedge n_{/\Gamma_R}$, where $u_e$ is the solution of the exterior problem $(P^e_{\omega,\beta})$.*

**Remark 2.1** *Note that we can define the operator $T_R(\omega,\beta)$ whatever choice of the geometry of the bounded domain $\Omega_i$, provided it is regular enough and contains the heart of the fiber. However, we have chosen to limit the domain of computation, with a disc of radius $R$, in order to express explicitly the operator $T_R(\omega,\beta)$ (see Appendix A). We give this expression in the following theorem.*

At this step it is useful to use polar coordinates $(r,\theta)$ in the $(x_1,x_2)$ plane and corresponding representations of $3D$ vector fields as follows:

$$\vec{u}(r,\theta) \ = \ u_r(r,\theta)\vec{e}_r + u_\theta(r,\theta)\vec{e}_\theta + u_3(r,\theta)\vec{e}_3, \tag{2.6}$$

where $(\vec{e}_r, e_\theta, \vec{e}_3)$ denotes the usual local orthonormal basis of $\mathbb{R}^3$ associated to the cylindrical coordinates (see figure 2.1). We shall make use also of the expansion in Fourier series with respect to the angular coordinate $\theta$. More precisely, we have, for any $\varphi \in \mathcal{D}'(\Gamma_R)$:

$$\left\{ \begin{array}{l} \varphi \ = \ \displaystyle\sum_{n\in\mathbb{Z}} \varphi^n e^{in\theta} \quad (\text{in } \ \mathcal{D}'(\Gamma_R)) \\ \text{where } \varphi^n = <\varphi, \dfrac{1}{2\pi R} e^{in\theta} >_{\mathcal{D}'(\Gamma_R),\mathcal{D}(\Gamma_R)}. \end{array} \right. \tag{2.7}$$

This permits us to characterize, for $s \in \mathbb{R}$, the Sobolev space $H^s(\Gamma_R)$, which is an Hilbert space as follows:

$$H^s(\Gamma_R) = \{\varphi \in \mathcal{D}'(\Gamma_R) \ ; \ \|\varphi\|^2_{H^s(\Gamma_R)} = 2\pi R \sum_{n\in\mathbb{Z}} (1+n^2)^s \ |\varphi^n|^2 < \infty\}. \tag{2.8}$$

We remarks also that, if the vector field $u = (\boldsymbol{u}, u_3) \in H(\text{rot},\Omega_i) \times H^1(\Omega_i)$, its tangential traces on $\Gamma_R$ have the following expressions:

$$\left\{ \begin{array}{rcl} \boldsymbol{u} \wedge n_{/\Gamma_R} & = & -u_\theta(R,\theta) \\ u \wedge n_{/\Gamma_R} & = & (u_3(R,\theta), -u_\theta(R,\theta)) = u_3\vec{e}_\theta - u_\theta \vec{u}_3 \\ (u\wedge n) \wedge n_{/\Gamma_R} & = & (-u_\theta(R,\theta), -u_3(R,\theta)) = -u_\theta\vec{e}_\theta - u_3\vec{u}_3 \end{array} \right. \tag{2.9}$$

**Theorem 2.1** *Let $\phi$ be a tangential vector field to $\Gamma_R$ in the space $H^{-\frac{1}{2}}(rot_\beta,\Gamma_R)$. We can write:*

$$\phi = -\phi_\theta \ e_\theta - \phi_3 \ e_3 \quad \textit{where} \quad \phi_\theta \in H^{-\frac{1}{2}}(\Gamma_R) \quad \textit{and} \quad \phi_3 \in H^{\frac{1}{2}}(\Gamma_R).$$

*and*

$$T_R(\omega,\beta)(\phi) \ = \ (T_R(\omega,\beta)(\phi))_\theta \ e_\theta \ + \ (T_R(\omega,\beta)(\phi))_3 \ e_3$$

*Moreover, using the Fourier expansion of $\phi$, we can express the operator $T_R(\omega,\beta)(\phi)$ as:*

$$\left\{ \begin{array}{c} (T_R(\omega,\beta)(\phi))_\theta \\ \\ (T_R(\omega,\beta)(\phi))_3 \end{array} \right\} = \sum_{n\in\mathbb{Z}} [T^n_R(\omega,\beta)] \left\{ \begin{array}{c} -\phi^n_\theta \\ \\ -\phi^n_3 \end{array} \right\} e^{in\theta}. \tag{2.10}$$

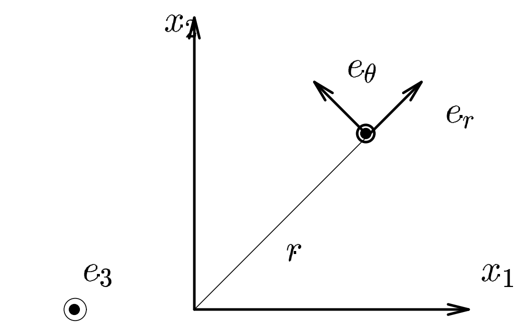



Figure 2.1: Cylindrical coordinates

*where*

$$T_R^n(\omega,\beta) = \begin{bmatrix} \alpha\dfrac{K_n(\alpha R)}{K_n'(\alpha R)} & \dfrac{i\beta n}{\alpha R}\dfrac{K_n(\alpha R)}{K_n'(\alpha R)} \\ -\dfrac{i\beta n}{\alpha R}\dfrac{K_n(\alpha R)}{K_n'(\alpha R)} & \dfrac{\beta^2-\alpha^2}{\alpha}\dfrac{K_n'(\alpha R)}{K_n(\alpha R)} - \dfrac{\beta^2 n^2}{\alpha^3 R^2}\dfrac{K_n(\alpha R)}{K_n'(\alpha R)} \end{bmatrix} \tag{2.11}$$

*and where*

$$\alpha = \sqrt{\beta^2 - \frac{\omega^2}{c_\infty^2}}.$$

*The operator $T_R(\omega,\beta)$ is symmetric and one has the following expression:*

$$\begin{cases} \forall(\phi,\psi) \in H^{-\frac{1}{2}}(rot_\beta,\Gamma_R)^2 \\ < T_R(\omega,\beta)(\phi),\psi >_{/\Gamma_R} = 2\pi R \displaystyle\sum_{n\in\mathbb{Z}} [T_R^n(\omega,\beta)] \begin{Bmatrix} \phi_\theta^n \\ \phi_3^n \end{Bmatrix} \cdot \begin{Bmatrix} \bar{\psi_\theta^n} \\ \bar{\psi_3^n} \end{Bmatrix}. \end{cases} \tag{2.12}$$

**Proof** The proof of the symmetry of $T_R(\omega,\beta)$ is straightforward by using Green's Formula and problem $(P^e_{\omega,\beta})$. It also appears on Formula (2.12) (note that each matrix $T_R^n(\omega,\beta)$ is hermitian), which is a consequence of (2.10), via Plancherel's theorem. For the proof of (2.11), we refer the reader to appendix A.

♦

### 2.2.1 The bilinear form $b_R(\omega,\beta)$

In order to establish a variational form of the interior problem, we will need to define, from the operator $T_R(\omega,\beta)$ the bilinear form denoted $b_R(\omega,\beta)$ as follows:

**Definition 2.2** *We define the symmetric bilinear form $b_R(\omega,\beta)$ on $H(rot_\beta,\Omega_i)\times H(rot_\beta,\Omega_i)$ by*

$$b_R(\omega,\beta)(u,v) = < T_R(\omega,\beta)((u\wedge n)\wedge n_{/\Gamma_R}),\,(v\wedge n)\wedge n_{/\Gamma_R} >$$

**Remark 2.2** *The continuity of $b_R(\omega,\beta)$ stems up from the one of $T_R(\omega,\beta)$ as well as the one of the trace mapping $\gamma_\tau$.*

From Theorem 2.1, we deduce an explicit formula for $b_R(\omega,\beta)$:

$$b_R(\omega,\beta)(u,v) = 2\pi R \sum_{n\in\mathbb{Z}} [T_R^n(\omega,\beta)] \begin{Bmatrix} \boldsymbol{u}_\theta^n \\ u_3^n \end{Bmatrix} \cdot \begin{Bmatrix} \bar{\boldsymbol{v}_\theta^n} \\ \bar{v_3^n} \end{Bmatrix} \tag{2.13}$$

where the matrices $[T_R^n(\omega,\beta)]$ are defined in (2.11), and where $\boldsymbol{u}_\theta^n$ et $u_3^n$ are respectively the $n^{th}$ coefficient of the Fourier expansion of the tangential trace $\boldsymbol{u}_\theta$ of $\boldsymbol{u}$ and the trace of $u_3$ on $\Gamma_R$. This is the formula which will be used for the numerical computation. One of the essential points of the study, is to obtain some coercivity results, related to the new interior problem we shall deal with. In this study, the sign of the bilinear form $b_R(\omega,\beta)$ plays a priori a fundamental role. This is the object of the following lemma, which proves that $b_R(\omega,\beta)$ is neither positive, nor negative.

**Lemma 2.4** *Let $t^{n-}(\omega,\beta) < t^{n+}(\omega,\beta)$ be the eigenvalues of the matrix $T_R^n(\omega,\beta)$. Then $t^{n-}(\omega,\beta) < 0$ $\forall n \in \mathbb{Z}$ (see figure 2.2) and the sequences $t^{n+}(\omega,\beta)$ and $t^{n-}(\omega,\beta)$ have the following asymptotic behaviour:*

$$
\begin{aligned}
t^{n+}(\omega,\beta) &\sim \frac{n}{R} \quad n \to +\infty \\
t^{n-}(\omega,\beta) &\sim \left(\frac{\omega^2}{c_\infty^2} - 2\beta^2\right)\frac{R}{n} \quad n \to +\infty.
\end{aligned}
$$

**Proof** Setting $\mathbb{K}_n = \dfrac{K_n'(\alpha R)}{K_n(\alpha R)}$, the computation of the eigenvalues of the matrices $T_R^n(\omega,\beta)$ gives:

$$
\left\{
\begin{aligned}
t^{n+}(\omega,\beta) &= \frac{1}{2}\mathbb{K}_n\left[\left(\frac{\omega^2}{\alpha c_\infty^2} + \left(\alpha - \frac{\beta^2 n^2}{\alpha^3 R^2}\right)(\mathbb{K}_n)^{-2}\right)\right. \\
&\quad \left. - \sqrt{\left\{\frac{-\omega^2}{\alpha c_\infty^2} + \left(\alpha + \frac{\beta^2 n^2}{\alpha^3 R^2}\right)(\mathbb{K}_n)^{-2}\right\}^2 + 4\frac{\beta^2 n^2}{\alpha^2 R^2}(\mathbb{K}_n)^{-4}}\right] \\
t^{n+}(\omega,\beta)t^{n-}(\omega,\beta) &= \frac{\omega^2}{c_\infty^2} - 2\frac{\beta^2 n^2}{\alpha^2 R^2}(\mathbb{K}_n)^{-2}.
\end{aligned}
\right.
$$

We skip here the proof of $t^{n-}(\omega,\beta) < 0$, which is technical, but not difficult.
Thanks to the asymptotic properties of the modified Bessel functions $K_n$ (see M. Abramowitz et I. Stegun [1] and chapter 7 of the thesis of H. Picq [24]), we have

$$
-\mathbb{K}_n = \frac{n}{\alpha R}\left[1 + \frac{(\alpha R)^2}{2n^2}(1 + o(1))\right] \tag{2.14}
$$

from which we deduce that

$$
t^{n+}(\omega,\beta) \sim -\alpha\mathbb{K}_n \quad \text{and} \quad t^{n+}(\omega,\beta)t^{n-}(\omega,\beta) \sim \frac{\omega^2}{c_\infty^2} - 2\beta^2.
$$

Then, it is immediate to conclude.

♦

We remark that $t^{n+}(\omega,\beta) > 0$ for $n$ large enough, which proves that $b_R(\omega,\beta)(u,u)$ has no sign. In particular it is not positive because of the negative eigenvalues $t^{n-}(\omega,\beta)$. This is an essential difference with the scalar case, treated in [14](in this case, the Dirichlet to Neumann map, which replaces the operator $T_R(\omega,\beta)$, is positive) and is also one of the sources of the main difficulties of the mathematical analysis (see section 2.4). In fact, we can even prove that the unbounded operator $B_R(\omega,\beta)$ from $L^2(\Omega_i)^3$ associated to the bilinear form $b_R(\omega,\beta)$ is not bounded from below. Indeed , we have the

**Lemma 2.5** *There exists a sequence $(u_p)$ in $V_\beta(\Omega_i)$, satisfying*

$$
\|u_p\|_{0,\Omega_i} = 1 \quad \text{and} \quad b_R(\omega,\beta)(u_p,u_p) \xrightarrow[p \to +\infty]{} -\infty.
$$

**Proof** Let us introduce a cut-off function $\varphi(r)$ satisfying

$$
\left\{
\begin{aligned}
&\varphi \in \mathcal{C}^\infty(\mathbb{R}^+) \qquad \text{Supp } \varphi \subset [0, 1/2] \\
&\varphi(r) \equiv 1 \text{ in the neighborhood of } r = 0.
\end{aligned}
\right.
$$

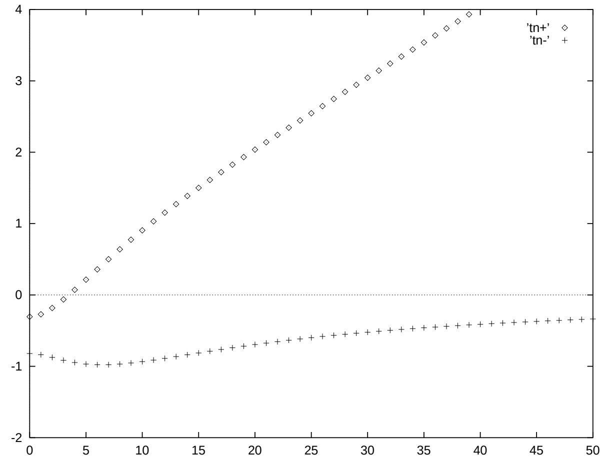



Figure 2.2: Eigenvalues of $T_R^n(\omega,\beta)$ with $(\omega,\beta) = (0.5, 1.)$, $c_\infty = 1.$ and $R = 10$

Then, we set: $u^p = p\ \varphi(p^2(1-\frac{r}{R}))\ \vec{e_\theta}$. The idea is to construct a vector field tangential to the boundary $\Gamma_R$ which, when $p$ increases, becomes more and more concentrated near $\Gamma_R$. One remarks that

$$\begin{aligned}\|u_p\|^2_{0,\Omega_i} &= 2\pi p^2 \int_0^R \varphi^2(p^2(1-\frac{r}{R}))r\ dr = 2\pi R^2 \int_0^{+\infty} \varphi^2(t)(1-\frac{t}{p^2})\ dt \\ &\xrightarrow[p\to+\infty]{} 2\pi R^2 \int_0^{+\infty} \varphi^2(t)\ dt > 0,\end{aligned}$$

whereas $b_R(\omega,\beta)(u_p,u_p) = 2\pi R\alpha \dfrac{K_0(\alpha R)}{K_0'(\alpha R)}\ p^2$. We conclude easily, since $\dfrac{K_0(\alpha R)}{K_0'(\alpha R)} < 0.$

♦

## 2.3 The interior problem in $V_\beta(\Omega_i)$

The transmission problem considered in Lemma 2.1 leads us naturally to introduce the following interior problem, in $\Omega_i$:

$$(\widetilde{P}^i_{\omega,\beta}) \quad \left\{ \begin{array}{l} \text{Find } (\omega^2, u_i) \in \mathbb{R}^{\star+} \times H(\mathrm{rot}_\beta, \Omega_i)\backslash\{0\} \\[1ex] \quad (2.15) \quad \mathrm{rot}^\star_\beta(\mu^{-1}\mathrm{rot}_\beta u_i) = \omega^2 \varepsilon u_i \quad \text{in } \Omega_i \\[1ex] \quad (2.16) \quad \mathrm{rot}_\beta u_i \wedge n_{/\Gamma_R} = T_R(\omega,\beta)((u_i\wedge n)\wedge n_{/\Gamma_R}). \end{array} \right.$$

The problem $(\widetilde{P}^i_{\omega,\beta})$ is equivalent to our original problem $(P)$ in the sense of the following theorem.

**Theorem 2.2** *Assume* $0 < \dfrac{\omega^2}{c_\infty^2} < \beta^2$

- *Let* $(\omega^2, u)$ $(u \neq 0)$ *be a solution of* $(P)$, *then* $(\omega^2, u_i := u_{/\Omega_i})$ *is a solution of* $(\widetilde{P}^i_{\omega,\beta})$.
- *Conversely, let* $(\omega^2, u_i)$ *be a solution of* $(\widetilde{P}^i_{\omega,\beta})$, *then* $(\omega^2, u_i)$ *has a unique extension to* $\mathbb{R}^2$, *solution of* $(P)$.

**Proof** Let $(\omega^2, u)$ be a solution of $(P)$. The restriction of $u$ to $\Omega_i$, $u_i$, is necessarily non zero since, if not, due to the transmission relations (2.1), $u_e := u_{/\Omega_e}$ would satisfy $(P^e_{\omega,\beta})$ with the boundary

condition $(u_e \wedge n) \wedge n_{/\Gamma_R} = 0$. The exterior $u_e$ would be zero everywhere and therefore $u$ would be also zero. Moreover, from the construction of the operator $T_R(\omega, \beta)$ and using once more the transmission relations (2.1), we deduce that $u_i$ satisfies the boundary condition (2.16).
Conversely, let $(\omega^2, u_i)$ be a pair solution of $(\widetilde{P}^i_{\omega,\beta})$. Let us set $u_e$ the unique solution of $(P^e_{\omega,\beta})$ associated to $(u_i \wedge n) \wedge n_{/\Gamma_R}$. Then, the fields $u_i$ and $u_e$ satisfy respectively $(P^i)$ and $(P^e)$. From the other hand, by construction of $u_e$, one has

$$(u_e \wedge n) \wedge n_{/\Gamma_R} \;=\; (u_i \wedge n) \wedge n_{/\Gamma_R}$$

$$\text{and} \left\{ \begin{aligned} \mathrm{rot}_\beta u_i \wedge n_{/\Gamma_R} \;&=\; T_R(\omega, \beta)((u_i \wedge n) \wedge n_{/\Gamma_R}) \\ &=\; T_R(\omega, \beta)((u_e \wedge n) \wedge n_{/\Gamma_R}) \\ &=\; \mathrm{rot}_\beta u_e \wedge n_{/\Gamma_R} \quad \text{(by definition of } T_R(\omega,\beta)). \end{aligned} \right.$$

Since the transmission relations (2.1) are satisfied, Lemma 2.1 shows that the field $u$, whose restrictions to $\Omega_i$ and $\Omega_e$ are respectively $u_i$ and $u_e$, is solution of $(P)$. The uniqueness of the extension $u$ is deduced immediately from the uniqueness of the solution of $(P^e_{\omega,\beta})$ when $\varphi_T = (u_i \wedge n) \wedge n_{/\Gamma_R}$.

♦

We give a variational form of this theorem, that is useful for numerical computations, in the following lemma. The proof is immediate using Green's formula (1.14) and the definition of $b_R(\omega, \beta)$.

**Lemma 2.6** *The pair $(\omega^2, u_i)$ is solution of $(\widetilde{P}^i_{\omega,\beta})$ if and only if*

$$\left\{ \begin{aligned} & (\omega^2, u_i) \in \mathbb{R}^{\star +} \times H(rot_\beta, \Omega_i) \\ & \int_{\Omega_i} \mu^{-1} rot_\beta u_i \cdot rot_\beta v_i \, dx \;+\; \mu_\infty^{-1} b_R(\omega, \beta)(u_i, v_i) \\ & = \omega^2 \int_{\Omega_i} \varepsilon u_i \cdot v_i \, dx \quad \forall v_i \in H(rot_\beta, \Omega_i) \end{aligned} \right. \tag{2.17}$$

Since we are looking for solutions satisfying the generalized free divergence condition, it is interesting to give a new version of Lemma 2.6, where we restrict ourselves to the space of the free divergence fields. Moreover this is necessary, if we want to get any type of compactness result. Similarly to the orthogonal decomposition (1.16) of $H_\varepsilon(\Omega_i)$, we first note that, since $H_\beta(\Omega_i)^\perp$ (see (1.16)) is included in $H(\mathrm{rot}_\beta, \Omega_i)$, we have also

$$\begin{aligned} H(\mathrm{rot}_\beta, \Omega_i) \;&=\; V_\beta(\Omega_i) \;\oplus\; H_\beta(\Omega_i)^\perp \\ \text{where} \quad V_\beta(\Omega_i) &= \{\, u_i \in H(\mathrm{rot}_\beta, \Omega_i) \;;\; \mathrm{div}_\beta(\varepsilon u_i) = 0 \,\}. \end{aligned} \tag{2.18}$$

This will be useful for proving the following lemma:

**Lemma 2.7** *The pair $(\omega^2, u_i)$ is solution of $(\widetilde{P}^i_{\omega,\beta})$ if and only if*

$$\left\{ \begin{aligned} & (\omega^2, u_i) \in \mathbb{R}^{\star +} \times V_\beta(\Omega_i) \;/\; a_{\omega,\beta}(u_i, v_i) = \omega^2 (u_i, v_i)_\varepsilon \quad \forall v_i \in V_\beta(\Omega_i) \\ & \textit{where } \; a_{\omega,\beta}(u_i, v_i) \;=\; \int_{\Omega_i} \mu^{-1} rot_\beta u_i \cdot rot_\beta v_i \, dx \;+\; \mu_\infty^{-1} b_R(\omega, \beta)(u_i, v_i) \end{aligned} \right. \tag{2.19}$$

**Proof** We only have to prove that any solution of (2.19) is solution of $(\widetilde{P}^i_{\omega,\beta})$. For this, thanks to the orthogonal decomposition (2.18) of $H(\mathrm{rot}_\beta, \Omega_i)$, we only need to show that

$$a_{\omega,\beta}(u_i, v_i^\perp) = \omega^2 \; (u_i, v_i^\perp)_\varepsilon \quad \forall (u_i, v_i^\perp) \in V_\beta(\Omega_i) \times H_\beta^\perp(\Omega_i).$$

But, from one hand $\forall v_i^\perp \in H_\beta^\perp(\Omega_i)$ $\quad \mathrm{rot}_\beta v_i^\perp = 0$ and $v_i^\perp \wedge n_{/\Gamma_R} = 0$, then

$$(\mu^{-1}\mathrm{rot}_\beta u_i, \mathrm{rot}_\beta v_i^\perp) = 0 \quad \text{and} \quad b_R(\omega,\beta)(u_i, v_i^\perp) = 0,$$

while, from the other hand, since $u_i \in H_\beta(\Omega_i)$, $\forall v_i^\perp \in H_\beta^\perp(\Omega_i)$ $\quad (u_i, v_i^\perp)_\varepsilon = 0$.

We associate to this new bilinear form, an unbounded operator $\widetilde{A_{\omega,\beta}}$ defined on $H_\beta(\Omega_i)$, such that

$$\left\{\begin{array}{ll} \mathcal{D}(\widetilde{A_{\omega,\beta}}) = \{\ u_i \in V_\beta(\Omega_i)\ ; & \mathrm{rot}_\beta^\star(\mu^{-1}\mathrm{rot}_\beta u_i) \in L^2(\Omega_i)^3; \\ & \mathrm{rot}_\beta u_i \wedge n_{/\Gamma_R} = T_R(\omega,\beta)((u_i\wedge n)\wedge n_{/\Gamma_R})\ \} \\ \widetilde{A_{\omega,\beta}} \ = \ \varepsilon^{-1}\mathrm{rot}_\beta^\star(\mu^{-1}\mathrm{rot}_\beta u_i) & \forall u_i \in \mathcal{D}(\widetilde{A_{\omega,\beta}}). \end{array}\right. \tag{2.20}$$

Lemma 2.7 expresses that:

| |
|---|
| *there exists a guided mode associated to $(\omega,\beta)$, with $0 < \omega < \beta c_\infty$, if and only if $\omega^2$ is an eigenvalue of $\widetilde{A_{\omega,\beta}}$.* |

In some sense, we have transformed our original problem set in $\mathbb{R}^2$ into a fixed point eigenvalue problem set in a bounded domain. However, we are going to see in the next paragraph that the spectral properties of $\widetilde{A_{\omega,\beta}}$ are difficult to analyze. In particular, it is not clear at all that $\widetilde{A_{\omega,\beta}}$ is self-adjoint, with a pure point spectrum, **which is in fact the ideal situation we are looking for**.

## 2.4 The difficulties in the analysis of $\widetilde{A_{\omega,\beta}}$

By construction, the operator $\widetilde{A_{\omega,\beta}}$ is clearly symmetric but it is not clear at all that it is self-adjoint! Such a property would be true if one would be able to obtain a Garding's type inequality of the form:

$$\left\{\begin{array}{l} \exists\lambda \in \mathbb{R} \quad \exists\alpha \in \mathbb{R}^{\star+} \text{ such that } \forall u_i \in V_\beta(\Omega_i) \\ a_{\omega,\beta}(u_i,u_i) \ + \ \lambda\|u_i\|_\varepsilon^2 \ \geq \ \alpha\|u_i\|^2_{H(\mathrm{rot}_\beta,\Omega_i)} \end{array}\right. \tag{2.21}$$

The only difficulty would consist in showing that, $\exists\alpha \in \mathbb{R}$ such that

$$b_R(\omega,\beta)(u_i,u_i) \ \geq \ \alpha\|u_i\|^2_{H(\mathrm{rot},\Omega_i)} - \lambda\|u_i\|^2_\varepsilon \quad \forall u_i \in V_\beta(\Omega_i).$$

We remark already that we can not hope this inequality with $\alpha = 0$. To localize the difficulty, let us present some computations: first, we express for each $u_i \in V_\beta(\Omega_i)$, $b_R(\omega,\beta)(u_i,u_i)$ with respect to $u_e \in H(\mathrm{rot}_\beta,\Omega_e)$, solution of $(P^e_{\omega,\beta})$ with $\varphi_T = (u_i \wedge n)\wedge n_{/\Gamma_R}$. We have:

$$\begin{aligned} b_R(\omega,\beta)(u_i,u_i) \quad &= < T_R(\omega,\beta)(u_i\wedge n)\wedge n_{/\Gamma_R}, (u_i\wedge n)\wedge n_{/\Gamma_R} > \\ &= \int_{\Omega_e} |\mathrm{rot}_\beta u_e|^2\, dx - \frac{\omega^2}{c_\infty^2}\int_{\Omega_e}|u_e|^2\, dx. \end{aligned}$$

Now, as for the proof of Lemma 2.3,

$$\int_{\Omega_e} |\mathrm{rot}_\beta u_e|^2\, dx = \int_{\Omega_e} (|\nabla u_{e3}|^2 + \beta^2|\boldsymbol{u_e}|^2 + |rot\boldsymbol{u_e}|^2)\, dx - 2\beta\int_{\Omega_e} \boldsymbol{u_e}\cdot\nabla u_{e3}\, dx$$

and

$$\begin{aligned} -2\beta\int_{\Omega_e} \boldsymbol{u_e}\cdot\nabla u_{e3}\, dx \quad &= 2\beta\int_{\Omega_e} div\boldsymbol{u_e}\, u_{e3}\, dx + 2\beta < \boldsymbol{u_e}\cdot n, u_{e3} >_{/\Gamma_R} \\ &= 2\beta^2\int_{\Omega_e}|u_{e3}|^2\, dx + 2\beta < \boldsymbol{u_e}\cdot n, u_{e3} >_{/\Gamma_R}, \end{aligned}$$

because $div\boldsymbol{u_e} = \beta u_{e3}$. We obtain on this way

$$b_R(\omega,\beta)(u_i,u_i) = \left.\begin{array}{c} \|rot\boldsymbol{u_e}\|^2_{0,\Omega_e} + (\beta^2 - \dfrac{\omega^2}{c^2_\infty})\|\boldsymbol{u_e}\|^2_{0,\Omega_e} \\ +(2\beta^2 - \dfrac{\omega^2}{c^2_\infty})\|u_{e3}\|^2_{0,\Omega_e} + \|\nabla u_{e3}\|^2_{0,\Omega_e} \end{array}\right\} > 0 \tag{2.22}$$
$$+2\beta < \boldsymbol{u_e}\cdot n, u_{e3} >_{/\Gamma_R} .$$

It would remain to estimate the boundary term $| < \boldsymbol{u_e}\cdot n, u_{e3} >_{/\Gamma_R} |$. We have

$$\begin{aligned} | < \boldsymbol{u_e}\cdot n, u_{e3} > /\Gamma_R| &\leq \|\boldsymbol{u_e}\cdot n\|_{H^{-\frac12}(\Gamma_R)} \|u_{e3}\|_{H^{\frac12}(\Gamma_R)} \\ &\leq C_R(\gamma)\, \|\boldsymbol{u_e}\cdot n\|_{H^{-\frac12}(\Gamma_R)} \|u_{e3}\|_{H^1(\Omega_e)}. \end{aligned} \tag{2.23}$$

The quantity $\|u_{e3}\|_{H^1(\Omega_e)}$ can be controlled by the positive part of $b_R(\omega,\beta)(u_i,u_i)$ (cf. (2.22)) via the use of Young's inequality. The difficulty comes from the term $\|\boldsymbol{u_e}\cdot n\|_{H^{-\frac12}(\Gamma_R)}$, on which we have no control. That is why we look for a new formulation in a bounded domain based on the construction of a new space $\widetilde{V}_\beta(\Omega_i)$ such that

(i) The variational formulation (2.19), where we replace $V_\beta(\Omega_i)$ by $\widetilde{V}_\beta(\Omega_i)$ remains equivalent to the problem $(\widetilde{P}^i_{\omega,\beta})$.

(ii) The bilinear form $a_{\omega,\beta}$ is coercive on $\widetilde{V}_\beta(\Omega_i)$.

(iii) $\widetilde{V}_\beta(\Omega_i) \underset{\text{compact}}{\hookrightarrow} L^2(\Omega_i)^3$.

Property (ii) will permit us to ensure the self-adjoint character of the new operator associated to $a_{\omega,\beta}$ in $\widetilde{V}_\beta(\Omega_i)$ (of course, it is important to underline that the operator associated to $a_{\omega,\beta}$ changes when we change the domain of definition of $a_{\omega,\beta}$).

Property (iii) will imply that this operator has a compact resolvent and thus a pure point spectrum. This is very important from the numerical point of view.

For both Property (ii) and (iii), it will appear that a new boundary condition on the normal trace of the field is needed. The interest for (ii) of such a new formulation has already been emphasized in the attempt of proof of (2.21). For (iii), let us recall that compactness results for vector fields require the control of both rotational and divergence. However, the injection from the space $H(\mathrm{rot}_\beta,\Omega_i) \cap H_\beta(\Omega_i)$ into $H_\beta(\Omega_i)$ is not compact. This is due to the fact that we have no control neither on the normal trace nor the tangential one of the field of this space. For instance, let us mention Ch. Weber results [31]:

$$H(div_\varepsilon,\Omega_i) \cap H_0(rot,\Omega_i) \underset{\text{compact}}{\hookrightarrow} L^2(\Omega_i)^2$$

$$H_0(div_\varepsilon,\Omega_i) \cap H(rot,\Omega_i) \underset{\text{compact}}{\hookrightarrow} L^2(\Omega_i)^2.$$

For these two results, the boundary condition plays an important role. Finally, the numerical modeling will point out the need of a new boundary condition on $\Gamma_R$ (see section 5).

# 3 Towards a second formulation

*In this section, we introduce a new trace operator $S_R(\omega,\beta)$, which allow us to establish a boundary condition on $\Gamma_R$ satisfied by each solution of $(\widetilde{P}^i_{\omega,\beta})$. With the help of this new condition, we construct a new space $V_{\omega,\beta}(\Omega_i)$, for which we show that it is compactly embedded in $L^2(\Omega_i)^3$.*

## 3.1 The operator $S_R(\omega,\beta)$ and its properties

Let us start to remark that, each solution $u$ of the initial problem $(P)$ satisfies $\mathrm{div}_\beta(\varepsilon u) = 0$ in $\mathbb{R}^2$. Therefore, the restrictions $u_i$ and $u_e$ to $\Omega_i$ and $\Omega_e$ verify:

$$\begin{cases} \mathrm{div}_\beta(\varepsilon u_i) = 0 & \text{in } \Omega_i \\ \mathrm{div}_\beta(\varepsilon u_e) = 0 & \text{in } \Omega_e \\ (3.1) \quad \varepsilon \boldsymbol{u_i}\cdot n \;=\; \varepsilon \boldsymbol{u_e}\cdot n & \text{on } \Gamma_R \end{cases}$$

We are going to take into account this additional transmission relation, which did not appear in the construction of the interior problem $(\widetilde{P}^i_{\omega,\beta})$ (see Section 2.3). We will construct $\boldsymbol{u_e}\cdot n_{/\Gamma_R}$ as a function of $(u_e\wedge n)\wedge n_{/\Gamma_R}$ (when $\beta^2 - \dfrac{\omega^2}{c_\infty^2} \;>\; 0$), in the same way that we constructed $\mathrm{rot}_\beta u_e \wedge n_{/\Gamma_R}$ as a function of $(u_e\wedge n)\wedge n_{/\Gamma_R}$, thanks to the exterior problem $(P^e_{\omega,\beta})$ (and more precisely with the help of the operator $T_R(\omega,\beta)$).

**Definition 3.1** *Let $\omega$ and $\beta$ be two strictly positive real numbers satisfying*

$$\beta^2 - \frac{\omega^2}{c_\infty^2} \;>\; 0.$$

*We denote by $S_R(\omega,\beta)$ the mapping from $H^{-\frac12}(rot_\beta,\Gamma_R)$ to $H^{-\frac12}(\Gamma_R)$ which associates to $\varphi_T$ the normal trace $\boldsymbol{u_e}\cdot n_{/\Gamma_R}$ where $u_e$ is the solution of the exterior problem $(P^e_{\omega,\beta})$.*

The boundary $\Gamma_R$ being a circle of radius $R$, we can obtain an explicit expression of the operator $S_R(\omega,\beta)$, in the same way as we did it for $T_R(\omega,\beta)$.

**Theorem 3.1** *Let be $\phi \in H^{-\frac12}(rot_\beta,\Gamma_R)$, then according to the notations of Theorem 2.1, we have*

$$S_R(\omega,\beta)(\phi) \;=\; \sum_{n\in\mathbb{Z}} S_R^n(\omega,\beta)\cdot\phi^n\, e^{in\theta}. \tag{3.2}$$

*where*

$$\phi^n \;=\; (-\phi^n_\theta, -\phi^n_3)$$

*and*

$$S_R^n(\omega,\beta) = \left( \frac{in}{\alpha R}\frac{K_n(\alpha R)}{K_n'(\alpha R)} \quad , \quad \frac{\beta}{\alpha}\,(\frac{n^2}{\alpha^2R^2}\frac{K_n(\alpha R)}{K_n'(\alpha R)} - \frac{K_n'(\alpha R)}{K_n(\alpha R)}\,) \right). \tag{3.3}$$

*Moreover, the operator $S_R(\omega,\beta)\in\mathcal{L}(H^{-\frac12}(rot_\beta,\Gamma_R), H^{-\frac12}(\Gamma_R))$.*

**Proof**
- The explicit computation of $u_e$, solution of $(P^e_{\omega,\beta})$ associated to $\phi$ has been done in Appendix A. The normal trace $\boldsymbol{u_e}\cdot n_{/\Gamma_R}$ is nothing but the radial component of the field $u_e$ in $(r=R)$, whose expression in Fourier expansion is given by (A.13).
- $\|S_R(\omega,\beta)(\phi)\|_{H^{-\frac12}(\Gamma_R)} \;=\; \|\boldsymbol{u_e}\cdot n\|_{H^{-\frac12}(\Gamma_R)}$.
By trace theorem, since $\mathrm{div}_\beta(u_e)=0$, we have:

$$\|\boldsymbol{u_e}\cdot n\|_{H^{-\frac12}(\Gamma_R)} \;\le\; C_R\; sup(1,\beta)\;\|u_e\|_{0,\Omega_e}.$$

The continuity of $S_R(\omega,\beta)$ comes from that of the mapping which associates to $\phi\in H^{-\frac12}(\mathrm{rot}_\beta,\Gamma_R)$ the field $u_e\in H(\mathrm{rot}_\beta,\Omega_e)\hookrightarrow L^2(\Omega_e)^3$ (cf. Lemma 2.2).

♦

In the sequel, we will set

$$\left\{\begin{array}{lcl} a^n &=& \dfrac{n}{\alpha R}\dfrac{K_n(\alpha R)}{K_n'(\alpha R)} \\ b^n &=& \dfrac{\beta}{\alpha}\,(\dfrac{n^2}{\alpha^2R^2}\dfrac{K_n(\alpha R)}{K_n'(\alpha R)} - \dfrac{K_n'(\alpha R)}{K_n(\alpha R)}\,), \end{array}\right. \tag{3.4}$$

in such a way that

$$S_R^n(\omega,\beta) \;=\; (\; i a^n \;,\; b^n \;). \tag{3.5}$$

Let us remark that $a^{-n} = -a^n$ and $b^{-n} = b^n \quad \forall n \in \mathbb{Z}$. Moreover, we shall need the following result.

**Lemma 3.1** *The real numbers $a^n$ and $b^n$ satisfy for each $n \in \mathbb{Z}^\star$,*

$$na^n \;<\; 0 \quad \textit{and} \quad b^n \;>\; -\frac{\beta}{\alpha}\frac{K_n(\alpha R)}{K_n'(\alpha R)}) \;>\; 0 \tag{3.6}$$

*and moreover have the following asymptotic behaviour*

$$a^n \;\sim_{n\to+\infty}\; -1 \quad , \quad b^n \;\sim_{n\to+\infty}\; \frac{\beta R}{n}. \tag{3.7}$$

**Proof** We deduce directly the relations (3.6) from the following inequalities

$$\forall n \in \mathbb{Z} \quad \forall z \in \mathbb{R}^{\star+} \qquad \left\{\begin{array}{l} K_n(z) \;>\; 0 \qquad K_n'(z) \;<\; 0 \\ 1 + \dfrac{n^2}{z^2} - [\dfrac{K_n'(z)}{K_n(z)}]^2 \;<\; 0. \end{array}\right. \tag{3.8}$$

From the asymptotic behaviour of $K_n'(\alpha R)/K_n(\alpha R)$ (2.14), we obtain at first

$$a^n \;\sim_{n\to+\infty}\; -1.$$

We deduce also that

$$\frac{n^2}{(\alpha R)^2}\frac{K_n(\alpha R)}{K_n'(\alpha R)} \;=\; -\frac{n}{\alpha R}\,[1 \;+\; \frac{(\alpha R)^2}{2n^2}(1+o(1)]^{-1}. \tag{3.9}$$

By addition of (2.14) and (3.9), we obtain

$$\begin{array}{lcl} b^n &=& (\dfrac{\beta}{\alpha})(\dfrac{n}{\alpha R})\{\;[1+\dfrac{(\alpha R)^2}{2n^2}+o(\dfrac{1}{n^2})] \;-\; [1-\dfrac{(\alpha R)^2}{2n^2}+o(\dfrac{1}{n^2})]\;\} \\ &=& \dfrac{\beta}{\alpha}(\dfrac{n}{\alpha R})\{\;\dfrac{(\alpha R)^2}{n^2}+o(\dfrac{1}{n^2})\;\} \;=\; \dfrac{\beta}{\alpha}(\dfrac{\alpha R}{n})(\;1 \;+\; o(1)\;). \end{array}$$

♦

**Remark 3.1** *The asymptotic behaviour of the vectors $S_R^n(\omega,\beta)$, when $n \to +\infty$, allows us to show directly the continuity of the operator $S_R(\omega,\beta)$ (cf. [26]).*

Now, we are going to define the operator $S_R^{inv}(\omega,\beta)$ which is a right-inverse of the operator $S_R(\omega,\beta)$. The interest in of this operator will appear in the section 3.2.

**Lemma 3.2** *The operator $S_R^{inv}(\omega,\beta)$ from $H^{-\frac{1}{2}}(\Gamma_R)$ into $H^{-\frac{1}{2}}(rot_\beta,\Gamma_R)$ defined by*

$$S_R^{inv}(\omega,\beta)(\varphi) = \sum_{n\in\mathbb{Z}} \left[\begin{array}{c} i\alpha^n \\ \\ \gamma^n \end{array}\right] \varphi^n e^{in\theta} \tag{3.10}$$

*where* $\begin{cases} \varphi^n \text{ is the } n^{th} \text{ Fourier coefficient of } \varphi \\ \gamma^n \ = \ (b^n - \dfrac{n}{\beta R} a^n)^{-1} > 0 \quad , \quad \alpha^n \ = \ \dfrac{n}{\beta R}\gamma^n, \end{cases}$

*is continuous. Moreover, it is a right-inverse of the operator* $S_R(\omega,\beta)$*: more precisely,* $\forall(\varphi,\psi) \in H^{-\frac{1}{2}}(\Gamma_R) \times H^{-\frac{1}{2}}(rot_\beta, \Gamma_R)$ *we have*

$$S_R^{inv}(\omega,\beta)(\varphi) \ = \ \psi \quad \Longrightarrow \quad \varphi \ = \ S_R(\omega,\beta)(\psi). \tag{3.11}$$

**Proof** Let us prove, at first that the operator $S_R^{inv}(\omega,\beta)$ applies, in a continuous way, $H^{-\frac{1}{2}}(\Gamma_R)$ into $H^{-\frac{1}{2}}(\mathrm{rot}_\beta, \Gamma_R)$. First, we deduce from the properties on $a^n$ and $b^n$ (3.6), that $b^n - \dfrac{n}{\beta \mathrm{R}} a^n > 0 \quad \forall n \in \mathbb{Z}$. From (3.7), we deduce that

$$\lim_{n \to +\infty} \alpha^n \ = \ 1. \tag{3.12}$$

Consequently,

$$\sum_{n\in\mathbb{Z}} (1+n^2)^{-\frac{1}{2}} |\alpha^n|^2 |\varphi^n|^2 \ < \ \text{Cste} \, \|\varphi\|^2_{H^{-\frac{1}{2}}(\Gamma_R)}, \tag{3.13}$$

and

$$\begin{aligned} \sum_{n\in\mathbb{Z}} (1+n^2)^{\frac{1}{2}} |\gamma^n|^2 |\varphi^n|^2 \ &= \ \sum_{n\in\mathbb{Z}} (1+n^2) |\gamma^n|^2 (1+n^2)^{-\frac{1}{2}} |\varphi^n|^2 \\ &= \ |\gamma^0|^2 |\varphi^0|^2 + \sum_{n\in\mathbb{Z}} (\frac{1}{n^2} + 1)(\beta R)^2 |\alpha^n|^2 (1+n^2)^{-\frac{1}{2}} |\varphi^n|^2 \\ &\le \ \text{Cste} \ \|\varphi\|^2_{H^{-\frac{1}{2}}(\Gamma_R)}, \end{aligned}$$

which proves that $S_R^{inv}(\omega,\beta)$ is defined and continuous from $H^{-\frac{1}{2}}(\Gamma_R)$ into $H^{-\frac{1}{2}}(\Gamma_R) \times H^{\frac{1}{2}}(\Gamma_R)$. Now, if $S_R^{inv}(\omega,\beta)(\varphi) = \psi = (-\psi_\theta, -\psi_3)$ then $S_R^n(\omega,\beta)(\psi^n) = (-\alpha^n a^n + \gamma^n b^n)\phi^n$ which shows that $S_R^{inv}(\omega,\beta)$ is a right-inverse of $S_R(\omega,\beta)$, since one checks easily that $\alpha^n$ and $\gamma^n$ are built in such a way that

$$-\alpha^n a^n \ + \ \gamma^n b^n \ = \ 1 \quad \forall n \in \mathbb{Z}.$$

♦

Thanks to the trace operator $S_R(\omega,\beta)$, we can introduce a new boundary condition on $\Gamma_R$.

**Lemma 3.3** *If* $(\omega^2, u_i)$ *is a solution of* $(\widetilde{P}^i_{\omega,\beta})$ *then the field* $u_i$ *satisfies the boundary condition:*

$$\boldsymbol{u_i} \cdot n_{/\Gamma_R} \ = \ S_R(\omega,\beta)((u_i \wedge n) \wedge n_{/\Gamma_R}) \tag{3.14}$$

**Proof** From the equivalence between the two problems $P$ and $(\widetilde{P}^i_{\omega,\beta})$ (cf. Theorem 2.2), the field $u_i$ solution of $(\widetilde{P}^i_{\omega,\beta})$ admits an unique extension to $\mathbb{R}^2$, $u$, solution of $P$, and whose restriction to $\Omega_e$ denoted $u_e$ is solution of the problem $(P^e_{\omega,\beta})$ such that $(u_e \wedge n) \wedge n_{/\Gamma_R} = (u_i \wedge n) \wedge n_{/\Gamma_R}$. Therefore by definition of $S_R(\omega,\beta)$, we have

$$S_R(\omega,\beta)((u_i \wedge n) \wedge n_{/\Gamma_R}) \ = \ \boldsymbol{u_e} \cdot n_{/\Gamma_R}.$$

The solution $u$ satisfying the free divergence_$\beta$ condition in the whole space $\mathbb{R}^2$, the transmission relation (3.1) allows us to conclude.

♦

Let us insist on the fact that this new boundary condition will be included in the functional space we are going to work with, contrary to the first one which appears directly in the variational formulation through $b_R(\omega,\beta)$.

## 3.2 The space $V_{\omega,\beta}(\Omega_i)$ and its properties

Therefore, let us introduce the space

$$V_{\omega,\beta}(\Omega_i) \;=\; \left\{ u_i \in V_\beta(\Omega_i) \;/\; \boldsymbol{u_i}\cdot n_{/\Gamma_R} \;=\; S_R(\omega,\beta)((u_i\wedge n)\wedge n_{/\Gamma_R}) \right\}, \tag{3.15}$$

which has the following interesting properties:

**Theorem 3.2**

(*i*) *As a closed subspace of* $H(rot_\beta,\Omega_i)$, $V_{\omega,\beta}(\Omega_i)$ *is an Hilbert space*

(*ii*) $V_{\omega,\beta}(\Omega_i) \underset{compact}{\hookrightarrow} L^2(\Omega_i)^3$

(*iii*) $V_{\omega,\beta}(\Omega_i)$ *is dense in* $H_\beta(\Omega_i)$.

**Remark 3.2** *This last density result is not so obvious as it seems to be because the space* $V_{\omega,\beta}(\Omega_i)$ *does not contain, for instance, the regular functions. In fact, even if the function* $\varepsilon$ *is very regular,* $V_{\omega,\beta}(\Omega_i)$ *will not contain all the regular functions, which do not necessarily satisfy the boundary condition appearing in (3.15). Our proof, which is really not trivial, is indirect and will involve an auxiliary operator* $B_{\omega,\beta}$ *defined with the help of* $S_R^{inv}(\omega,\beta)$ *and makes use Theorem 1.3.*

**Proof** **(i)** - Let us consider a sequence $v^n \in V_{\omega,\beta}(\Omega_i)$ converging towards $v$ in $H(\mathrm{rot}_\beta,\Omega_i)$. Obviously, $\mathrm{div}_\beta(\varepsilon v) = 0$ in $\Omega_i$. Moreover, by trace theorem and continuity of $S_R(\omega,\beta)$, we have:

$$S_R(\omega,\beta)((v^n\wedge n)\wedge n_{/\Gamma_R}) \;\longrightarrow\; S_R(\omega,\beta)((v\wedge n)\wedge n_{/\Gamma_R}) \text{ in } H^{-\frac{1}{2}}(\Gamma_R). \tag{3.16}$$

We have also:

$$\left\{ \begin{array}{l} div(\varepsilon\boldsymbol{v^n}) = \beta\varepsilon v_3^n \\ \\ div(\varepsilon\boldsymbol{v}) = \beta\varepsilon v_3 \end{array} \right. \;\Longrightarrow\; \varepsilon\boldsymbol{v^n} \;\to\; \varepsilon\boldsymbol{v} \text{ in } H(div,\Omega_i). \tag{3.17}$$

Therefore, by trace theorem in $H(\mathrm{div},\Omega_i)$, we can pass to the limit in $\boldsymbol{v^n}\cdot n_{/\Gamma_R} \;=\; S_R(\omega,\beta)((v^n\wedge n)\wedge n_{/\Gamma_R})$ to get:

$$\boldsymbol{v}\cdot n_{/\Gamma_R} \;=\; S_R(\omega,\beta)((v\wedge n)\wedge n_{/\Gamma_R}).$$

$V_{\omega,\beta}(\Omega_i)$ is therefore closed in $H(\mathrm{rot}_\beta,\Omega_i)$.

**(ii)** - Let $(u_i)^n$ be a sequence of $V_{\omega,\beta}(\Omega_i)$ such that

$$\|u_i^n\|_{H(\mathrm{rot}_\beta,\Omega_i)} \;\leq\; C.$$

To each $u_i^n$, let us associate $u_e^n$ the solution of $(P^e_{\omega,\beta})$ such that $(u_e^n\wedge n)\wedge n_{/\Gamma_R} \;=\; (u_i^n\wedge n)\wedge n_{/\Gamma_R}$. The field $u^n$ whose restrictions to $\Omega_i$ and $\Omega_e$ are respectively $u_i^n$ and $u_e^n$, satisfies the two following properties:

1) The sequence $(u^n)$ is included in $H(\mathrm{rot}_\beta,\mathbb{R}^2)$ and

$$\|u^n\|_{H(\mathrm{rot}_\beta,\mathbb{R}^2)} \;\leq\; \mathrm{Cste}. \tag{3.18}$$

Indeed, using well posedness of problem $(P_e^\omega)$ and a trace theorem:

$$\begin{aligned} \|u_e^n\|_{H(\mathrm{rot}_\beta,\Omega_e)} \;&\leq\; C\|(u_i^n\wedge n)n_{/\Gamma_R}\|_{H^{-\frac{1}{2}}(\mathrm{rot}_\beta,\Gamma_R)} \\ &\leq\; C\,C_R(\gamma_\tau)\|u_i^n\|_{H(\mathrm{rot}_\beta,\Omega_i)} \\ &\leq\; C\,C_R(\gamma_\tau)\,C \end{aligned}$$

2) $\mathrm{div}_\beta(\varepsilon u^n) = 0$. This assertion is deduced from the fact that

$$\left\{\begin{array}{lll} \mathrm{div}_\beta(\varepsilon u_i^n) = 0 \text{ in } \Omega_i & & \\ \mathrm{div}_\beta(\varepsilon u_e^n) = 0 \text{ in } \Omega_e & & \\ \boldsymbol{u_i}^n \cdot n_{/\Gamma_R} & = & S_R(\omega,\beta)((u_i^n \wedge n) \wedge n_{/\Gamma_R}) \\ & = & S_R(\omega,\beta)((u_e^n \wedge n) \wedge n_{/\Gamma_R}) \\ & = & \boldsymbol{u_e}^n \cdot n_{/\Gamma_R}. \end{array}\right.$$

Moreover, $\|div(\varepsilon \boldsymbol{u^n})\|^2_{0,\mathbb{R}^2} \;=\; \beta^2 \|\varepsilon u_3^n\|^2_{0,\mathbb{R}^2} \;\leq\; C$ (by (3.18)).

We deduce from these two points that $(\boldsymbol{u^n}) \in H(rot, \mathbb{R}^2) \cap H(div_\varepsilon, \mathbb{R}^2)$ and is bounded in this space. We can now recall the following compactness result, proved in [16]) using Ch. Weber's results [31]:

**Proposition 3.1** *Let us introduce the Hilbert space*

$$H(rot, div_\varepsilon, \mathbb{R}^2) = \{\boldsymbol{u} \in L^2(\mathbb{R}^2)^2 \;/\; rot\boldsymbol{u} \in L^2(\mathbb{R}^2)\;,\; div(\varepsilon\boldsymbol{u}) \in L^2(\mathbb{R}^2)\} \tag{3.19}$$

*equipped with the norm*

$$\| \boldsymbol{u} \|_\varepsilon^2 = \int_{\mathbb{R}^2} (| \boldsymbol{u} |^2 + | rot\boldsymbol{u} |^2 + | div(\varepsilon\boldsymbol{u}) |^2)\; dx. \tag{3.20}$$

*Then the mapping* $\boldsymbol{u} \to \boldsymbol{u}\,|_{B_R}$ *is compact from* $H(rot, div_\varepsilon, \mathbb{R}^2)$ *into* $L^2(B_R)^2$.

Therefore, from this compactness property, we can extract from $\boldsymbol{u^n}$ a subsequence $\boldsymbol{u^{n_k}}$ such that

$$\boldsymbol{u_i^{n_k}} \;=\; \boldsymbol{u^{n_k}}_{/\Omega_i} \;\longrightarrow\; \boldsymbol{u} \text{ in } L^2(\Omega_i)^2.$$

From the compact embedding of $H^1(\Omega_i)$ into $L^2(\Omega_i)$, once again from this subsequence $u_i^{n_k}$, we can extract another subsequence, still denoted $u_i^{n_k}$ such that

$$u_{i3}^{n_k} \;\longrightarrow\; u_3 \text{ in } L^2(\Omega_i),$$

which achieves the demonstration of (ii).

**(iii)** - The proof of this third part, which is technical, is given for the sake of clarity in Appendix B.

♦

## 3.3 The new problem

We can now define the new interior problem:

$$(P^i_{\omega,\beta}) \quad \left\{\begin{array}{l} \text{Find } (\omega^2, u_i) \in \mathbb{R}^{\star+} \times V_{\omega,\beta}(\Omega_i) \text{ such that} \\ a_{\omega,\beta}(u_i, v_i) = \omega^2 (u_i, v_i)_\varepsilon \quad \forall v_i \in V_{\omega,\beta}(\Omega_i). \end{array}\right.$$

We have shown that, if $u_i$ is solution of $(\widetilde{P}^i_{\omega,\beta})$, then $u_i$ belongs to $V_{\omega,\beta}(\Omega_i)$ (cf. Lemma 3.3). The space of test functions $V_{\omega,\beta}(\Omega_i)$ being included in $V_\beta(\Omega_i)$, $u_i$ is automatically solution of $(P^i_{\omega,\beta})$. The converse is not so obvious and will be proven in the next section. Finally, the problem $(P^i_{\omega,\beta})$ is the one we will discretize numerically in Section 5.

# 4 Analysis of problem $(P^i_{\omega,\beta})$

*We prove the equivalence of the two problems $(P^i_{\omega,\beta})$ and $(P)$, using the intermediate one $(\widetilde{P}^i_{\omega,\beta})$ (see Theorem 4.2) and a generalized Helmholtz decomposition (in Theorem 4.1). We then study the mathematical properties of the new problem, from which, we deduce a characterization of the dispersion relations of the guided modes (Theorem 4.4), and that we will approximate by a finite element method.*

## 4.1 Equivalence between the two problems $(P^i_{\omega,\beta})$ and $(\widetilde{P}^i_{\omega,\beta})$

The equivalence between the initial problem (P) stated on $\mathbb{R}^2$ and the problem $(\widetilde{P}^i_{\omega,\beta})$ has been already stated in Theorem 2.2. Note also that to show the equivalence between the problem $(\widetilde{P}^i_{\omega,\beta})$ and the variational formulation (2.19), we used an orthogonal decomposition (2.18) of $H(\mathrm{rot}_\beta, \Omega_i)$. Once again, the equivalence of the two problems $(\widetilde{P}^i_{\omega,\beta})$ and $(P^i_{\omega,\beta})$ will make an essential use of an adequate, but non orthogonal, decomposition of $H(\mathrm{rot}_\beta, \Omega_i)$.

**Theorem 4.1**

$$\left\{ \begin{array}{l} \forall u_i \in V_\beta(\Omega_i) \quad \exists (v_i, \phi) \in V_{\omega,\beta}(\Omega_i) \times H^1(\mathbb{R}^2) \quad \textit{such that} \\ \\ \qquad\qquad u_i \;=\; v_i \;+\; \nabla_\beta \phi_{/\Omega_i} \end{array} \right. \tag{4.1}$$

**Remark 4.1** *In fact, this decomposition holds also to each $u_i$ belonging to $H(rot_\beta, \Omega_i)$, since we have shown, thanks to the orthogonal decomposition (cf. Theorem 1.2) of $H_\beta(\Omega_i)$, that*

$$\left\{ \begin{array}{l} \forall u_i \in H(rot, \Omega_i) \quad \exists (v_i, \phi_i) \in V_\beta(\Omega_i) \times H^1_0(\Omega_i) \\ \\ \qquad\qquad u_i \;=\; v_i \;+\; \nabla_\beta \phi_i. \end{array} \right.$$

**Proof** See Appendix C.

♦

### 4.1.1 Equivalence between the problems $(P^i_{\omega,\beta})$ and $(\widetilde{P}^i_{\omega,\beta})$

**Theorem 4.2** $(P^i_{\omega,\beta})$ *and* $(\widetilde{P}^i_{\omega,\beta})$ *are equivalent.*

**Proof** From Lemma 3.3, each solution $u_i$ of $(\widetilde{P}^i_{\omega,\beta})$ belongs to $V_{\omega,\beta}(\Omega_i)$. It is thus a solution of problem $(P^i_{\omega,\beta})$ since $V_{\omega,\beta}(\Omega_i) \subset V_\beta(\Omega_i)$. Conversely, if

$$a_{\omega,\beta}(u_i, v_i) = \omega^2 (u_i, v_i)_{\varepsilon,\Omega_i} \quad \forall v_i \in V_{\omega,\beta}(\Omega_i), \tag{4.2}$$

we have to show that this equality still remains true for each $v_i \in V_\beta(\Omega_i)$. Indeed, from the generalized Helmholtz decomposition (4.1), it suffices to verify that

$$a_{\omega,\beta}(u_i, \nabla_\beta \phi) = \omega^2 (u_i, \nabla_\beta \phi)_{\varepsilon,\Omega_i} \quad \forall \phi \in H^1(\mathbb{R}^2). \tag{4.3}$$

Due to the property $\mathrm{rot}_\beta(\nabla_\beta \phi) = 0$, the expression $a_{\omega,\beta}(u_i, \nabla_\beta \phi)$ is reduced to $a_{\omega,\beta}(u_i, \nabla_\beta \phi) = \mu_\infty^{-1} b_R(\omega, \beta)(u_i, \nabla_\beta \phi)$ (cf. (2.19) ). If we set $u_e$ the solution of $(P^e_{\omega,\beta})$ with $\varphi_T = (u_i \wedge n) \wedge n_{/\Gamma_R}$, we have, with the help of Green's formula:

$$-b_R(\omega, \beta)(u_i, \nabla_\beta \phi) = \frac{\omega^2}{c_\infty^2} (u_e, \nabla_\beta \phi)_{0,\Omega_e}.$$

So we have

$$a_{\omega,\beta}(u_i, \nabla_\beta \phi) \; = \; -\omega^2 \varepsilon_\infty (u_e, \nabla_\beta \phi)_{0,\Omega_e}.$$

Green's formula (1.15) and the fact that $u_i \in V_{\omega,\beta}(\Omega_i)$, that is $\boldsymbol{u_i} \cdot n_{/\Gamma_R} = \boldsymbol{u_e} \cdot n_{/\Gamma_R}$, imply that

$$\begin{aligned}
-\varepsilon_\infty (u_e, \nabla_\beta \phi)_{0,\Omega_e} \; &= \; \varepsilon_\infty (\mathrm{div}_\beta u_e, \phi)_{0,\Omega_e} + < \varepsilon_\infty \boldsymbol{u_e} \cdot n, \phi >_{/\Gamma_R} \\
&= < \varepsilon_\infty \boldsymbol{u_e} \cdot n, \phi >_{/\Gamma_R} \\
&= < \varepsilon_\infty \boldsymbol{u_i} \cdot n, \phi >_{/\Gamma_R} \\
&= \; (\mathrm{div}_\beta(\varepsilon u_i), \phi)_{0,\Omega_i} + (\varepsilon u_i, \nabla_\beta \phi)_{0,\Omega_i} \\
&= \; (\varepsilon u_i, \nabla_\beta \phi)_{0,\Omega_i}
\end{aligned}$$

which gives the equality (4.3). ♦

## 4.2 The new operator $A_{\omega,\beta}$ and its properties

The space $V_{\omega,\beta}(\Omega_i)$ being dense in $H_\beta(\Omega_i)$, we can associate to the bilinear form $a_{\omega,\beta}$ the unbounded operator $A_{\omega,\beta}$ of $H_\beta(\Omega_i)$, defined by:

$$\left\{ \begin{aligned}
&\mathcal{D}(A_{\omega,\beta}) = \;\; \{ \, u_i \in V_{\omega,\beta}(\Omega_i) \; ; \; \mathrm{rot}^\star_\beta(\mu^{-1} \mathrm{rot}_\beta u_i) \in L^2(\Omega_i)^3 \\
&\qquad\qquad \mathrm{rot}_\beta u_i \wedge n_{/\Gamma_R} = T_R(\omega,\beta)((u_i \wedge n) \wedge n_{/\Gamma_R}) \, \} \\
&A_{\omega,\beta} u_i \; = \; \varepsilon^{-1} \mathrm{rot}^\star_\beta(\mu^{-1} \mathrm{rot}_\beta u_i) \quad \forall u_i \in \mathcal{D}(A_{\omega,\beta}).
\end{aligned} \right. \tag{4.4}$$

The following theorem allows us to characterize the spectrum of the operator $A_{\omega,\beta}$.

**Theorem 4.3** *The bilinear form $a_{\omega,\beta}$ is coercive on $V_{\omega,\beta}(\Omega_i)$, that is $\exists \alpha > 0$ et $\exists C > 0$ such that*

$$\forall u \in V_{\omega,\beta}(\Omega_i) \quad a_{\omega,\beta}(u,u) + \alpha \|u\|^2_{\varepsilon,\Omega_i} \geq C \, \|u\|^2_{H(rot_\beta,\Omega_i)}. \tag{4.5}$$

**Proof** Let us consider $u_i \in V_{\omega,\beta}(\Omega_i)$. We proved in Section 2.4 ( cf. (2.22) and (2.23) ) that $u_e$ being solution of $P^e_{\omega,\beta}$ associated to the boundary condition $(u_e \wedge n) \wedge n_{/\Gamma_R}) = (u_i \wedge n) \wedge n_{/\Gamma_R})$, then we had

$$\begin{aligned}
b_R(\omega,\beta)(u_i,u_i) \; = \; & \|rot\boldsymbol{u_e}\|^2_{0,\Omega_e} + (\beta^2 - \frac{\omega^2}{c^2_\infty}) \|\boldsymbol{u_e}\|^2_{0,\Omega_e} \\
& + (2\beta^2 - \frac{\omega^2}{c^2_\infty}) \|u_{e3}\|^2_{0,\Omega_e} + \|\nabla u_{e3}\|^2_{0,\Omega_e} \\
& + 2\beta < \boldsymbol{u_e} \cdot n, u_{e3} >_{/\Gamma_R}
\end{aligned}$$

where $\quad | < \boldsymbol{u_e} \cdot n, u_{e3} >_{/\Gamma_R} > | \; \leq \; C_R \, \|\boldsymbol{u_e} \cdot n\|_{H^{-\frac{1}{2}}(\Gamma_R)} \, \|u_{e3}\|_{H^1(\Omega_e)}$.

But, in this new case, we can conclude since $u_i \in V_{\omega,\beta}(\Omega_i)$ i.e. $\boldsymbol{u_i} \cdot n_{/\Gamma_R} \; = \; \boldsymbol{u_e} \cdot n_{/\Gamma_R}$ (cf. the definition of $S_R(\omega,\beta)$). Then by trace theorem and since $\mathrm{div}(\varepsilon u_i) = \beta \varepsilon u_{i3}$:

$$\begin{aligned}
\|\varepsilon_\infty \boldsymbol{u_i} \cdot n\|_{H^{-\frac{1}{2}}(\Gamma_R)} \; &\leq \; \tilde{C_R} \, (\|\varepsilon \boldsymbol{u_i}\|^2_{0,\Omega_i} + \|div(\varepsilon \boldsymbol{u_i})\|^2_{0,\Omega_i})^{\frac{1}{2}} \\
&\leq \; \tilde{C_R} \, (\|\varepsilon \boldsymbol{u_i}\|^2_{0,\Omega_i} + \|\beta \varepsilon u_{i3}\|^2_{0,\Omega_i})^{\frac{1}{2}} \\
&\leq \; \tilde{C_R} \sqrt{\varepsilon^+} \max(1,\beta) \, \|u_i\|_{\varepsilon,\Omega_i}.
\end{aligned}$$

Therefore, if we set $C(\beta,\varepsilon) = \beta\tilde{C_R} C_R \sqrt{\varepsilon^+}\varepsilon_\infty^{-1}\max(1,\beta)$

$$2\beta| < \boldsymbol{u_e}\cdot n, u_{e3} >_{/\Gamma_R} > | \;\leq\; 2C(\beta,\varepsilon)\, \|u_i\|_{\varepsilon,\Omega_i}\; \|u_{e3}\|_{H^1(\Omega_e)}.$$

By Young's inequality, we obtain $\quad \forall \eta > 0$

$$\begin{aligned} b_R(\omega,\beta)(u_i,u_i) \;&\geq\; (2\beta^2 - \frac{\omega^2}{c_\infty^2} - C(\beta,\varepsilon)\eta)\|u_{e3}\|^2_{0,\Omega_e} \\ &-\frac{C(\beta,\varepsilon)}{\eta}\|u_i\|^2_{\varepsilon,\Omega_i} + (1 - C(\beta,\varepsilon)\eta)\|\nabla u_{e3}\|^2_{0,\Omega_e}. \end{aligned}$$

Choosing the parameter $\eta > 0$ such that $\eta\, C(\beta,\varepsilon) \;<\; \min(1, 2\beta^2 - \dfrac{\omega^2}{c_\infty^2})$, we obtain

$$b_R(\omega,\beta)(u_i,u_i) \;\geq\; -\frac{C(\beta,\varepsilon)}{\eta}\|u_i\|^2_{\varepsilon,\Omega_i}, \tag{4.6}$$

from it, we deduce that

$$a_{\omega,\beta}(u_i,u_i) + \frac{2\varepsilon_\infty^{-1}C(\beta,\varepsilon)}{\eta}\|u_i\|^2_{\varepsilon,\Omega_i} \;\geq\; \|\mathrm{rot}_\beta u_i\|^2_{0,\Omega_i} + \frac{\varepsilon_\infty^{-1}C(\beta,\varepsilon)}{\eta}\|u_i\|^2_{\varepsilon,\Omega_i}, \tag{4.7}$$

which proves the coercivity of $a_{\omega,\beta}$.

♦

**Corollary 4.1** *The operator $A_{\omega,\beta}$ is self-adjoint, bounded from below, with a compact resolvent. Its spectrum is then a pure point spectrum and is characterized by*

$$\sigma(A_{\omega,\beta}) \;=\; \{\; \lambda_n(\omega,\beta)\;;\; n \in \mathbb{N}\;\} \tag{4.8}$$

*where $\lambda_n(\omega,\beta)$ is an increasing sequence tending to $+\infty$*

**Proof** The corollary is deduced from the following results: The bilinear form $a_{\omega,\beta}$ is symmetric, coercive on $V_{\omega,\beta}$ (cf. Theorem 4.3). The space $V_{\omega,\beta} \underset{\text{compact}}{\hookrightarrow} H_\beta(\Omega_i)$ (cf. Theorem 3.2).

♦

Finally, we can characterize a guided mode, as follow:

**Theorem 4.4** *The triplet $(u,\omega,\beta)$ where the real numbers $\beta > 0$ and $\omega > 0$ satisfy $\beta^2 - \dfrac{\omega^2}{c_\infty^2} \;>\; 0$, represents a guided mode if and only if*

$$\left\{ \begin{array}{l} (\omega,\beta) \textit{ is solution of one of the equations: } \lambda_n(\omega,\beta) \;=\; \omega^2\;,\; n \in \mathbb{N} \\[2ex] \textit{and } u_i = u_{/\Omega_i} \textit{ is an eigenvector of } A_{\omega,\beta} \textit{ associated to the eigenvalue } \lambda_n(\omega,\beta). \end{array} \right. \tag{4.9}$$

**Remark 4.2** *By limiting the domain of the section $\mathbb{R}^2$ to a bounded domain $\Omega_i$, the study of the point spectrum of $A_\beta$ amounts to the one of the pure point spectrum of the operator $A_{\omega,\beta}$, which is an important step from the numerical point of view. However we introduce an additional non linearity in the problem, since $\omega$ appears in the same time, as a parameter of $A_{\omega,\beta}$ and as an eigenvalue of this operator.*

# 5 Numerical method and results

*We have solved the essential difficulty, from both conceptual and theoretical point of view, by reducing our problem to a bounded domain, thanks to Theorem 4.4. This result suggests an algorithm with two steps for the computation of the guided modes:*

*(i) Step 1: Computation of the eigenvalues $\lambda_n(\omega,\beta), n \geq 0$ of the operator $A_{\omega,\beta}$, the pair $(\omega,\beta)$ being given and satisfying $0 < \omega^2 < c_\infty^2\beta^2$.*

*(ii) Step 2: Resolution by a fixed point method of the equations:*

$$\lambda_n(\omega,\beta) = \omega^2 \quad \beta c_- < \omega < \beta c_\infty \quad n \geq 0$$

*The difficult step of the algorithm is the first one, which requires a numerical approximation by finite elements, which is the aim of this section. The mathematical analysis of the method, we will introduce in this work, is delayed to a forthcoming study. Nevertheless, we will give some conjectures, based on the study of the closed waveguides (see [17]) and also on some recent results of the literature. Some numerical results presented in the section 6, essentially allow us to validate our method. In the sequel, we will only work in the interior computation domain $\Omega_i = B_R$. In order to relieve the notations, we will suppress all the references to the domain $\Omega_i$: the space $V_\beta(\Omega_i)$ is denoted by $V_\beta$ and so on.*

## 5.1 Resolution by a finite element method of the eigenvalues of $A_{\omega,\beta}$

In a classical way, our objective is to define an approximate operator defined in a finite dimensional space to precise and whose eigenvalues are the approximations of the eigenvalues of $A_{\omega,\beta}$. In practice, we will have to solve the eigenvalues of a symmetric matrix. Let us notice here that we have to introduce two parameters of approximation

- h: step-size of the mesh of $\Omega_i = B_R$ for the spatial discretization, devoted to tend towards 0,
- N: series truncation parameter, devoted to tend to $+\infty$, which interferes implicitly, in the computation of the boundary contribution of the bilinear form $a_{\omega,\beta}$.

Thanks to these two parameters, we will define:

- an approximation space $V^N_{\omega,\beta,h}$ of $V_{\omega,\beta}$, corresponding to a non-conforming finite element method:$V^N_{\omega,\beta,h} \not\subset V_{\omega,\beta}$
- an approximate bilinear form $a^N_{\omega,\beta}(u,v)$ defined on $V^N_{\omega,\beta,h}$. Then the approximate problem can be formulated as:

$$\left\{\begin{array}{l} \text{Find} \quad (u,\lambda) \in V^N_{\omega,\beta,h}\backslash\{0\} \times \mathbb{R}^{\star+} \\ a^N_{\omega,\beta}(u,v) \;=\; \lambda(u,v)_\varepsilon \quad \forall v \in V^N_{\omega,\beta,h}. \end{array}\right. \tag{5.1}$$

Of course, the eigenpairs $(u,\lambda)$ solutions of (5.1) will be, when $h$ and $N^{-1}$ will tend to 0, approximations of the eigenpairs of $A_{\omega,\beta}$ . For the clarity of the paper, we choose to split into two parts the approximation, dealing with successively:

(i) the space discretization associated to $h$

(ii) the truncation of the series associated to $N$

The step (i) will lead us to construct a first finite dimensional space $V_{\omega,\beta,h}$, and to allow us to obtain a first approximation problem, where the parameter $h$ will only appear. The step (ii) will lead us to slightly modify this space in a space $V_{\omega,\beta,h}^{N}$ (and also the bilinear form) where the both parameters $N$ and $h$ will appear jointly.

### 5.1.1 Spatial discretization

In the sequel, $\tau_h$ represents a "conforming triangulation " of the open set $\Omega_i = B_R$ in the following meaning: $\overline{\Omega_i} = \bigcup_{K\in\tau_h} K$ where $\{K\}$ are disjoint triangles, that may be curvilinear when two of their vertices are on the boundary $\Gamma_R$. We set $h = \max_{K\in\tau_h} \mathrm{diam}(K)$.

**Remark 5.1** *In a conceptual point of view, it is not forbidden to replace triangles by quadrangles (eventually curvilinear). Nevertheless, let us recall that even in the case of the closed waveguide ([17]), the theoretical analysis of the convergence of the method is only complete for the case of triangles, the discrete compactness result of Kikuchi [19], being valid only for triangles (to our knowledge).*

The triangulation $\tau_h$ being defined, we shall consider the same basic spaces as for the closed waveguides [17]:

- The space $P_h$, as an approximation of the space $H^1(\Omega_i)$, is the space of lower order Lagrangian finite elements [28], which are linear on each triangle of $\tau_h$ and bilinear on each rectangle. Any element of $P_h$ is continuous and is uniquely determined by its values at the vertices of $\tau_h$.
- We use the space $R_h$ denoting the space related to the lowest order mixed elements of Nedelec [22] for the approximation of $H(\mathrm{rot}, \Omega_i)$. In 2D, these elements are easily deduced from those of Raviart-Thomas [27] for the approximation of the space $H(\mathrm{div}, \Omega_i)$ (simply apply a rotation of angle $\frac{\pi}{2}$).

$$\begin{aligned} R_h \;&=\; \{\boldsymbol{v} \in H(\mathrm{rot}, \Omega_i)\ ; \\ &\boldsymbol{v}_{/K} \in \{\alpha + \gamma(-x_2, x_1)^t, \alpha \in P_0(K)^2, \gamma \in P_0(K)\} \text{ if K} \in \tau_h \text{ is a triangle} \\ &\boldsymbol{v}_{/K} \in Q_{0,1}(K) \times Q_{1,0}(K) \text{ if K} \in \tau_h \text{ is a rectangle}\big\}\,. \end{aligned} \tag{5.2}$$

Any element of $R_h$ has tangential components which are continuous across the edges of $\tau_h$. It is uniquely determined by the constant values of its tangential components on the edges.

In the sequel, we shall denote by $\mathcal{N}_I$ the number of vertices ($\in \mathcal{I}_h$) of the mesh $\tau_h$ and by $\mathcal{N}_A$ the number of its edges ($\in \mathcal{A}_h$, including the edges of the boundary). The spaces $P_h$ and $R_h$ have been defined for meshes consisting of triangles or rectangles and we will explain, how we will deal with meshes including curvilinear elements: for each curvilinear triangle $\tilde{K}$, we introduce the corresponding triangle $K$, defined by the tree vertices of $\tilde{K}$. We know in fact in a natural way how to define each function of $P_h$ or of $R_h$ on the triangle $K$. In order to extend it to $\tilde{K}$ we choose the unique analytic extension, in the present case polynomial since the restriction of each function of $P_h$ or $R_h$ to each triangle $K$ is itself polynomial. Let us note that in order to define the basis fields of the space $R_h$, it is more useful to introduce the edges of the "straight mesh" instead of those of the "curvilinear mesh". Thus, in the sequel, $\mathcal{A}_h$ will indicate the set of edges of the "straight mesh".

**Remark 5.2** *When we will make the calculations, we will replace the triangles $\tilde{K}$ by the $K$ one. This is intuitive that, when the step-size of the mesh goes to zero, this process will converge. We can also conjecture, since we work here with finite elements of lowest order, that the geometrical* *error estimate has the same order as the error estimate due to the discretization in the interior domain. In other words, we do not lose anything. But the question remains posed. It is clear that, if we would use finite elements of upper order, we should improve the approximation of the boundary, and use isoparametric finite elements for instance (let us cite M. Lenoir [20], P.G. Ciarlet [8]) in order to keep the same accuracy as in the interior domain.*

We can now define the space $V_{\omega,\beta,h}$, almost as in the case of the closed waveguide. Let us remind that we proved in [17] that it was necessary to take into account the free divergence condition in the weak sense (which induces the non-conforming nature of the finite element approximation). The main difference is due to the non local boundary condition:

$$\boldsymbol{u} \cdot n \;=\; S_R(\omega,\beta)((u \wedge n) \wedge n). \tag{5.3}$$

The space $V_{\omega,\beta,h}$ is therefore defined as the subspace of $R_h \times P_h$ of the vectors fields $u_h$ satisfying in a weak sense (see below) both this boundary condition (5.3) and the generalized free divergence_$\beta$ condition. In a more precise way, we see that the boundary condition (5.3) appears naturally, when we write $\mathrm{div}_\beta(\varepsilon u) \;=\; 0$ for a field $u = (\boldsymbol{u}, u_3)$ of $V_{\omega,\beta}$, in a weak sense. Indeed:

$$div_\beta(\varepsilon u) \;=\; 0 \quad \Longleftrightarrow \quad \beta \varepsilon u_3 \;=\; div(\varepsilon \boldsymbol{u}), \tag{5.4}$$

which is equivalent to:

$$\beta \int_{B_R} \varepsilon u_3 \varphi \, dx \;+\; \int_{B_R} \varepsilon \boldsymbol{u} \cdot \nabla\varphi \, dx \;=\; < \boldsymbol{u} \cdot n, \varphi >_{\Gamma_R} \quad \forall \varphi \in H^1(B_R) \tag{5.5}$$

that is to say, for $u \in V_{\omega,\beta}$, equivalent to

$$\beta \int_{B_R} \varepsilon u_3 \varphi \, dx \;+\; \int_{B_R} \varepsilon \boldsymbol{u} \cdot \nabla\varphi \, dx \;=\; s_R(\omega,\beta)(u,\varphi) \tag{5.6}$$

where $s_R(\omega,\beta)$ is the bilinear form associated to the operator $S_R(\omega,\beta)$ (cf. section 3.1), defined on the space $V_\beta \times H^1(\Omega_i)$ by

$$s_R(\omega,\beta)(u,\varphi) \;=\; < S_R(\omega,\beta)(u \wedge n) \wedge n, \varphi >_{\Gamma_R}, \tag{5.7}$$

that we are further led to split up in the sum of two bilinear forms $s^1_R(\omega,\beta)(u_3,\varphi)$ and $s^2_R(\omega,\beta)(\boldsymbol{u},\varphi)$ in order to distinguish the respective roles of the longitudinal component $u_3$ and the transverse field $\boldsymbol{u}$. The fundamental benefit of Formula (5.6) is the fact that it uses only terms which make sense for fields in $H(\mathrm{rot}) \times H^1_0$ and that can thus be easily approximated (This would not be the case with Formula (5.5) because of the normal trace $\boldsymbol{u} \cdot n$. This emphasizes the interest of the boundary condition (5.3) from the numerical point of view). We can now define the space $V_{\omega,\beta,h}$ as follows:

$$V_{\omega,\beta,h} \;=\; \{u_h = (\boldsymbol{u}_h, u_{3h}) \in R_h \times P_h \;\; \text{satisfying the relation 5.8} \;\}$$

$$\text{where } \; (5.8): \quad \forall \varphi_h \in P_h \quad \beta \oint_{B_R} \varepsilon u_{3,h} \varphi_h dx \;+\; \int_{B_R} \boldsymbol{u}_h \cdot \nabla\varphi_h dx \;=\; s_R(\omega,\beta)(u_h,\varphi_h)$$

where the symbol $\oint$ denotes a numerical integration formula as the one used for the case of the closed waveguide (see [17]):

$$\oint_{B_R} f dx \;=\; \sum_{K \in \tau_h} \frac{\mathrm{meas}(K)}{3} \sum_{M \text{ a vertex of } K} f(M), \tag{5.9}$$

and whose fundamental interest is mass lumping: the operator in $P_h$ associated to $\oint_{B_R} \varepsilon u_{3,h} \varphi_h dx$ is diagonal in the usual basis $\{\lambda_i\}$ of $P_h$ [28]. It is easy to see, that $V_{\omega,\beta,h}$ is isomorphic to $R_h$.

Indeed let us rewrite (5.8) as:

$$\left\{\begin{array}{l} \forall \varphi_h \in P_h \quad \beta \oint_{B_R} \varepsilon u_{3,h}\varphi_h dx + \int_{B_R} \varepsilon \boldsymbol{u}_h \cdot \nabla\varphi_h dx \;=\\ \qquad\qquad s_R^1(\omega,\beta)(u_{3,h},\varphi_h) + s_R^2(\omega,\beta)(\boldsymbol{u}_h,\varphi_h), \end{array}\right. \tag{5.10}$$

where we set (see Theorem 3.1 and Formula 3.4):

$$\left\{\begin{array}{rcl} s_R^1(\omega,\beta)(u_3,\varphi) &=& -2\pi R \displaystyle\sum_{n\in\mathbb{Z}} b^n \; u_3^n \overline{\varphi^n} \\ s_R^2(\omega,\beta)(\boldsymbol{u},\varphi) &=& -2i\pi R \displaystyle\sum_{n\in\mathbb{Z}} a^n \; \boldsymbol{u}_\theta^n \overline{\varphi^n} \end{array}\right. \tag{5.11}$$

$u_3^n$, $\boldsymbol{u}_\theta^n$ and $\varphi_n$ being respectively the $n^{th}$ Fourier coefficients of $u_{3/\Gamma_R}$, $-\boldsymbol{u}\wedge \boldsymbol{n}_{/\boldsymbol{\Gamma_R}}$, and of $\varphi_{/\Gamma_R}$. We show now

**Lemma 5.1** *The field $\boldsymbol{u}_h$ being given in $R_h$, there exists a unique element $u_{3,h} \in P_h$ which satisfies (5.10), we shall denote by $J_{3,h}\boldsymbol{u}_h$.*

**Proof** Indeed, if we write the equation (5.10) for $\varphi_h \;=\; \lambda_i$ where $\lambda_i$ is a basis function of $P_h$ associated to an <u>interior</u> vertex $i$, we simply obtain ($\lambda_{i/\Gamma_R} \;=\; 0$):

$$\beta \oint_{B_R} \varepsilon u_{3,h}\lambda_i dx \;+\; \int_{B_R} \varepsilon \boldsymbol{u}_h \cdot \nabla\lambda_i dx \;=\; 0,$$

which defines completely $u_{3,h}$ in the vertex $i$. It remains to compute $u_{3,h}$ in each vertex of the boundary. It is easy to note that we are led to invert a $N_b \times N_b$ dimensional matrix ($N_b$ representing the number of vertices of the mesh localized on the boundary $\Gamma_R$), whose $(i,j)$ coefficient is given by:

$$\beta \oint_{B_R} \varepsilon\lambda_i\lambda_j dx \;+\; 2\pi R \sum_{n\in\mathbb{Z}} b^n \lambda_i^n \overline{\lambda_j^n}, \qquad 1 \leq i,j \leq N_b \tag{5.12}$$

the second member having as $i^{th}$ component:

$$s_R^2(\omega,\beta)(\boldsymbol{u}_h,\lambda_i) \;-\; \int_{B_R} \varepsilon\nabla\lambda_i \cdot \boldsymbol{u}_h dx. \tag{5.13}$$

Therefore, it suffices to prove that the matrix defined by (5.12), we shall denote in the sequel by $\mathbb{S}_R^h(\omega,\beta)$, is symmetric positive definite. The matrix whose $(i,j)^{th}$ coefficient is $\beta \oint_{B_R} \varepsilon\lambda_i\lambda_j \; dx$ being diagonal, with strictly positive terms ($\beta > 0$), it then suffices to check that the symmetric matrix whose $(i,j)^{th}$ coefficient is: $2\pi R \sum_{n\in\mathbb{Z}} b^n \lambda_i^n \overline{\lambda_j^n}$ is also positive. Let $Q_R^h$ be the quadratic form associated to this matrix in $\mathbb{R}^{N_b}$, and if $\phi_h = (\phi_i)_{i=1}^{Nb}$ in $\mathbb{R}^{N_b}$, set $\varphi_h \;=\; \sum_{i\in N_b} \phi_i\lambda_i \in P_h$, then

$$(Q_R^h \phi_h, \phi_h) \;=\; 2\pi R \sum_{n\in\mathbb{Z}} b^n \; |\varphi_h^n|^2, \tag{5.14}$$

where $\varphi_h^n$ are the Fourier coefficients of $\varphi_h/_{\Gamma_R}$. Then the result is deduced from the positivity of the coefficients $b^n$ (cf. Lemma 3.1).

♦

So we have proved that the relation (5.10) defines an isomorphism, denoted by $J_h$, from $R_h$ in $V_{\omega,\beta,h}$, which associates to each $\boldsymbol{u}_h \in R_h$ the field $(\boldsymbol{u}_h, J_{3,h}\boldsymbol{u}_h)$. It follows that dim $V_{\omega,\beta,h} = \mathcal{N}_A$ and that we construct a basis of $V_{\omega,\beta,h}$ considering

$$w_A \;=\; (\boldsymbol{w}_A, J_{3,h}\boldsymbol{w}_A) \qquad A \in \mathcal{A}_h \tag{5.15}$$

where $\boldsymbol{w}_A$ is the usual basis field of $R_h$ associated to the edge $A$ ([22],[4]). It is instructive to give the structure of these basis functions, according to whether we consider an interior edge or a frontier one.
Case of an interior edge $A$: In this case, the basis function $w_A$ coincides with the one already described in the case of a closed waveguide (cf. [17]): The support of $\boldsymbol{w}_a$ is the union of the both triangles having the common edge $A$ whereas the support of the third component $J_{3,h}\boldsymbol{w}_A$ is larger and more precisely made of all triangles connected with these two ones (see figure 5.1.1).
Case of a frontier edge $A$: In this case, the support of $\boldsymbol{w}_A$ is the unique curvilinear triangle $\tilde{K}$ associated to the triangle $K$ containing the edge $A$. But on the other hand, the third component $J_{3,h}\boldsymbol{w}_A$ has a non local support, which is made of the union of the triangles connected to $\tilde{K}$ and of the ones having at least one vertex on the boundary $\Gamma_R$. This last property arises from the full structure of the matrix $\mathbb{S}_R^h(\omega,\beta)$, which is linked to the non local character of the operator $S_R(\omega,\beta)$. We illustrate this property on figure 5.1.1. The space $V_{\omega,\beta,h}$ being defined, we can

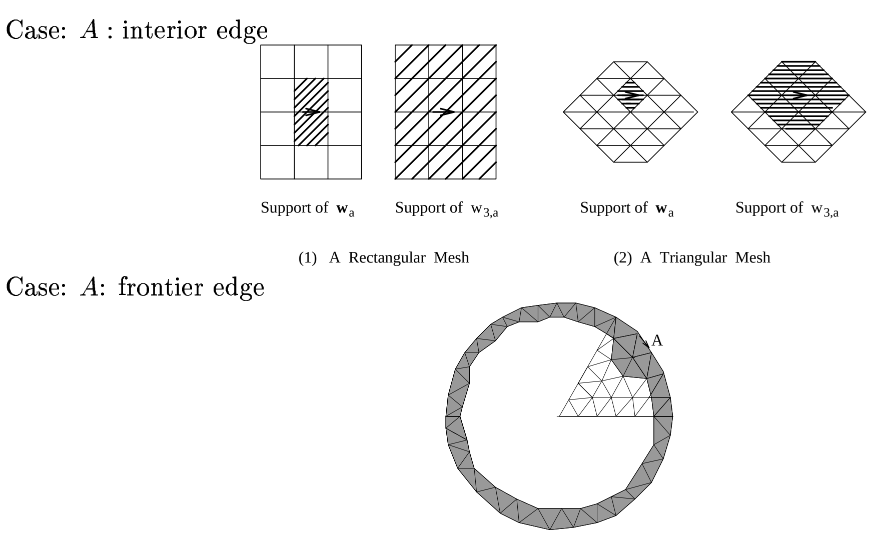



Figure 5.1: Support of the third component of $w_A$

formulate the approximation eigenvalue problem as follows:

$$\left| \begin{array}{l} \text{Find } \ (u_h,\lambda_h) \in V_{\omega,\beta,h}\backslash\{0\} \times \mathbb{R} \\ \\ a_{\omega,\beta}(u_h,v_h) \ = \ \lambda_h(u_h,v_h)_\varepsilon \quad \forall v_h \in V_{\omega,\beta,h}. \end{array} \right. \tag{5.16}$$

The space $V_{\omega,\beta,h}$ having a finite dimension, we associate to the symmetric bilinear form $a_{\omega,\beta}$ a self-adjoint operator $A_{\omega,\beta,h}$ in $V_{\omega,\beta,h}$, with the spectrum $\sigma(A_{\omega,\beta,h}) \ = \ \left\{\lambda_n^h(\omega,\beta), \quad 1 \le n \le \mathcal{N}_A\right\}$. Of course, the equations

$$\lambda_n^h(\omega,\beta) \ = \ \omega^2 \qquad 0 < \omega < \beta c_\infty \tag{5.17}$$

will define the dispersion equations of the numerical guided modes.

**Remark 5.3** *As we work with the matrix $\mathbb{A}_{\omega,\beta,h}$ whose $(a,b)$ element is given by: $\mathbb{A}_{\omega,\beta,h}(a,b) = a_{\omega,\beta}(w_a,w_b)$, the computation of the eigenvalues $\lambda_n^h(\omega,\beta)$ requires the following preliminary steps:*

*(i) The computation of the basis functions $w_a$, when $a$ belongs to $\Gamma_R$, which needs to invert the matrix $\mathbb{S}_R^h(\omega,\beta)$.*

*(ii) The computation of the terms $a_{\omega,\beta}(w_a,w_b)$.*

Of course, the essential non standard and delicate work concerns the computation of the matrices associated to the non local trace operators. These ones depending on infinite series, we will have to truncate these series in order to be able to carry out the computations. This justifies the second approximation (see section 5.1.2). Before doing it, we indicate a property, we wish to ask to the triangulation $\tau_h$, in order to reduce both the storage and the computations:

**Assumption 5.1** *All the triangles $K$ associated to a boundary curvilinear one $\tilde{K}$ are equal and isosceles.*

We notice that the assumption 5.1 involves that the curvilinear triangles connected to the boundary are isometric. This property is also true for the triangles connected to the boundary by one vertex. We enumerate the vertices of the boundary from 1 to $N_b$, in such a way that the vertices $j$ and $j+1$ belong to the same edge. Then we can show easily that, thanks to the assumption 5.1, the matrix $\mathbb{S}_R^h(\omega,\beta)$ is Toeplitz, that is:

$$\mathbb{S}_R^h(\omega,\beta)(a,b) \;=\; f(|a-b|) \quad \forall (a,b) \in \{1,..,N_b\}^2. \tag{5.18}$$

In other word, on each diagonal of the matrix, all elements are identical. In practice, we need only to compute and store $N_b$ elements instead of $N_b^2$. To invert $\mathbb{S}_R^h(\omega,\beta)$, we can also use a special algorithm inverting Toeplitz matrices in $O(N_b^2)$ operations (see [15]). We can also prove an analogous property for the matrix $\mathbb{B}_R(\omega,\beta)$ produced by the non local part $b_R(\omega,\beta)$ of the bilinear form $a_{\omega,\beta}$. We refer the reader to [26] for the details which are not really important. What we want to point out here is the influence of the geometric assumption 5.1 on the effective calculations and storage, which are then significantly reduced.

### 5.1.2 Series truncation

In this part, we denote by $N > 1$, the truncation parameter, which is intended to tend to $+\infty$. The first step consists in "approximating" the space $V_{\omega,\beta,h}$ by the space $V_{\omega,\beta,h}^N$, having the same dimension $\mathcal{N}_A$ and defined by:

$$\begin{aligned} & u_h \in V_{\omega,\beta,h}^N \;=\; \{u_h \in R_h \times P_h \;\text{ such that} \\ (5.19) \quad & \forall \varphi_h \in P_h \quad \beta \oint_{B_R} \varepsilon u_{3,h}\varphi_h dx \;+\; \int_{B_R} \boldsymbol{u}_h \cdot \nabla \varphi_h dx \;=\; s_R^N(\omega,\beta)(u_h,\varphi_h)\;\} \end{aligned}$$

$$\text{where} \quad \left| \begin{aligned} & s_R^N(\omega,\beta)(u,\varphi) \;=\; s_R^{1,N}(\omega,\beta)(u_3,\varphi) \;+\; s_R^{2,N}(\omega,\beta)(\boldsymbol{u},\varphi) \\ & s_R^{2,N}(\omega,\beta)(u_3,\varphi) \;=\; -2\pi R \sum_{|n|\le N} b^n u_3^n \,\overline{\varphi_n} \\ & s_R^{2,N}((\omega,\beta);\boldsymbol{u},\varphi) = 2i\pi R \sum_{|n|\le N} a^n \boldsymbol{u}_\theta^n \,\overline{\varphi_n}, \end{aligned} \right.$$

It is clear that each new basis function of $V_{\omega,\beta,h}^N$ associated to an interior edge is also a basis function of $V_{\omega,\beta,h}$. It is easy to see that (5.19) defines once again a new isomorphism $J_h^N$ from $R_h$ into $V_{\omega,\beta,h}^N$. Only the basis functions associated to the frontier edges are modified. More precisely , only the computation of $w_{3,A}$, where the edge $A$ is on the boundary, changes: we are led to invert the $N_b$ dimensional linear system, whose second member has the following $i^{\text{th}}$ component:

$$s_R^{2,N}(\omega,\beta)(\boldsymbol{u}_h,\lambda_i) \;-\; \int_{B_R} \varepsilon \nabla \lambda_i \cdot \boldsymbol{u}_h \, dx,$$

and whose component $(i,j)$ of the matrix denoted $\mathbb{S}_R^{h,N}(\omega,\beta)$ is defined by

$$\beta \oint_{B_R} \varepsilon \lambda_i \cdot \lambda_j \, dx \;+\; s_R^{1,N}(\omega,\beta)(\lambda_i,\lambda_j).$$

We check easily that the positive definite character of the matrix $\mathbb{S}_R^{h,N}(\omega,\beta)$ is preserved from the truncation of the series, because this one arises from the positivity of the coefficients $\boldsymbol{b}^n$ (see Lemma 3.1).

The space $V_{\omega,\beta,h}^N$ being constructed, we write the new discretized eigenvalue problem in the variational form, as follows:

$$\left\{\begin{array}{l} \text{Find } (u_h^N, \lambda_h^N) \in V_{\omega,\beta,h}^N \backslash\{0\} \times \mathbb{R} \text{ such that} \\[2ex] a_{\omega,\beta}^N(u_h^N, v_h) \;=\; \lambda_h^N\,(u_h^N, v_h)_\varepsilon \quad \forall v_h \in V_{\omega,\beta,h}^N \end{array}\right. \tag{5.20}$$

where the bilinear form $a_{\omega,\beta}^N$ is defined on the space $H(\text{rot}) \times H(\text{rot})$ by:

$$\begin{array}{l} a_{\omega,\beta}^N(u,v) \;=\; \displaystyle\int_{B_R} \text{rot}_\beta u \cdot \text{rot}_\beta v \; dx \;+\; b_R^N(\omega,\beta)(u,v) \\[2ex] \text{where } \; b_R^N(\omega,\beta)(u,v) \;=\; 2\pi R \displaystyle\sum_{|n|\le N} T_R^n(\omega,\beta) \begin{pmatrix} \boldsymbol{u}_\theta^n \\[2ex] u_3^n \end{pmatrix} \cdot \begin{pmatrix} \overline{\boldsymbol{v}_\theta^n} \\[2ex] \overline{v_3^n} \end{pmatrix} \end{array} \tag{5.21}$$

where we recall that $u_3^n$ (respectively $u_\theta^n$) denotes the $n^{\text{th}}$ Fourier coefficient of the trace (respectively the tangential trace) of $u_3$ (respectively of $\boldsymbol{u}$) on the boundary $\Gamma_R$. As in Section 5.1.1, to the bilinear form $a_{\omega,\beta}^N(.,.)$, we associate a self-adjoint operator $A_{\omega,\beta,h}^N$ from $V_{\omega,\beta,h}^N$ into itself. Its eigenvalues denoted by $\lambda_n^{h,N}(\omega,\beta) \quad 1 \le n \le \mathcal{N}_A$, are approximations of the eigenvalues $\lambda_n(\omega,\beta)$ of the operator $A_{\omega,\beta}$. The dispersion relations of the numerical guided modes, or in other words the fixed point equations really resolved are the following:

$$\lambda_n^{h,N}(\omega,\beta) \;=\; \omega^2 \quad 0 < \omega < \beta c_\infty \quad 1 \le n \le \mathcal{N}_A, \tag{5.22}$$

which correspond to the effective implementation.

## 5.2 Comments about some numerical difficulties

We notice that the difficult step of the computation of the matrix $\mathbb{A}_{\omega,\beta,h}^N$ consists in evaluating the 'boundary matrix', which arises from the $b_R^N(\omega,\beta)$, whose expression is given by

$$b_R^N(\omega,\beta)(u,v) \;=\; 2\pi R \sum_{|n|\le N} T_R^n(\omega,\beta) \begin{pmatrix} u_\theta^n \\[2ex] u_3^n \end{pmatrix} \cdot \begin{pmatrix} \overline{v_\theta^n} \\[2ex] \overline{v_3^n} \end{pmatrix}.$$

First of all, it is necessary to compute the four coefficients composing the matrix $T_R^n(\omega,\beta)$. These coefficients depend essentially on $\text{LK}_n(\alpha R) \;=\; \dfrac{K_n'(\alpha R)}{K_n(\alpha R)}$, where $K_n$ is the modified Bessel function of second kind, of order $n$ and where $\alpha$ is equal to $\sqrt{\beta^2 - \omega^2/c_\infty^2}$. The numerical computation of the terms $\text{LK}_n(r)$ may become instable, when $n$ becomes large. In particular, it is dangerous to compute separately the terms $K_n(r)$ and $K_n'(r)$, because they increase exponentially with $n$, whereas their ratio increases only linearly. It appears more clever to compute them with the help of a recurrence relation, which avoids these instabilities. Due to the following properties of the functions $K_n$:

$$\forall n \in \mathbb{Z} \quad \forall z \in \mathbb{C} \quad \left\{\begin{array}{lcl} K_{n-1}(z) - K_{n+1}(z) &=& -\dfrac{2n}{z} K_n(z) \\[2ex] K_{n-1}(z) + K_{n+1}(z) &=& -2K_n'(z) \end{array}\right., \tag{5.23}$$

we establish this relation:

$$\mathrm{LK}_{n+1}(\alpha R) \;=\; \frac{1}{\mathrm{LK}_n(\alpha R) - \dfrac{n}{\alpha R}} \;-\; \frac{n+1}{\alpha R}. \tag{5.24}$$

These coefficients appear also in the coefficients of the matrix $\mathbb{S}_R^{h,N}(\omega,\beta)$.

## 5.3 Description of the numerical procedure

Contrary to what (cf. (5.22)) could lead us to believe, for a given value $\beta$, one has only a finite set of equations to solve, since (5.22) cannot have solution if $n$ is large enough. Also, we do not know if the solution of each equation (5.22) is unique as it is the case for the scalar problem (see [11] for a discussion of this point).

For the numerical solution of (5.22), as $\omega$ belongs to $]\beta c_-, \beta c_\infty[$, it is useful to consider the new unknown $V = \dfrac{\omega}{\beta}$ which varies in the interval $]c_-, c_\infty[$. We will solve in practice:

$$\lambda_n^{h,N}(\beta V, \beta) \;=\; \beta^2 V^2 \qquad c_- < V < c_\infty \tag{5.25}$$

for a small number of $n$'s. For each of them we have to use a fixed point algorithm.

**The fixed point algorithm** We have chosen a variant of the secant method, called Illinois algorithm. During the algorithm of solving the fixed point equations (5.25), we need to compute the eigenvalues $\lambda_n^{h,N}(\omega,\beta)$ of the matrix $\mathbb{A}^N_{\omega,\beta,h}$. In order to avoid the difficulty, due to the eventual multiple eigenvalues, we have chosen a block Lanczos algorithm (see B.N. Parlett, H. Simon and L.M. Stringer [23], B. Vital [30], D.C. Sorensen [29]).

**Remark 5.4** *If we want to draw the dispersion curves $\beta \to \omega(\beta)$, it is useful to couple the previous algorithm to a continuation method with respect to $\beta$ [26]. In such a case, one of the problems, consists in finding the starting point of the curves, that is to say the thresholds. A rigorous approach would consist in designing a numerical method for solving the thresholds equations. This will be treated in a future work. These equations have been determined in [16]. From another point of view, we can notice that formally the thresholds are the values corresponding to $\omega = c_\infty\beta$ and then "would be solutions" of $\lambda_n(c_\infty\beta,\beta) = c_\infty^2\beta^2$. Unfortunately, the function $\lambda_n(\omega,\beta)$ is not defined for $\omega = c_\infty\beta$ (that is why we need the thresholds equations). An intermediate approach consists in solving, for $\varepsilon > 0$ small enough, the equations:*

$$\lambda_n(c_\infty\beta - \varepsilon, \beta) = (c_\infty\beta - \varepsilon)^2.$$

## 5.4 Some remarks about the convergence of the method

There are two sources of error in the method: the discretization of the domain $B_R$ ($h \to 0$), the truncation of the series ($N \to +\infty$). Then, we can define two errors:

(i) error corresponding to the approximation of $\lambda_n(\omega,\beta)$ by $\lambda_n^{N,h}(\omega,\beta)$

(ii) error as we replace the fixed points of $\lambda_n(\omega,\beta)$ by these one of $\lambda_n^{N,h}(\omega,\beta)$.

Such a study has been developed by A.S. Bonnet Ben-Dhia and N. Gmati [11], for the model of scalar waves. We can insist on the point that even we have similar results for our problem, the technique of the proof is not so similar. It arises from properties of the scalar problem, which are lost or unknown in our case, that is:

- Results of monotony with respect to $(\omega,\beta)$ of the eigenvalues $\lambda_n(\omega,\beta)$ (unknown in our case)
- The fact that the bilinear form $b_R(\omega,\beta)$ is expressed with a series of positive terms (false here)

- The fact that the finite element approximation is conform (false here).

These properties allow to obtain some estimates, with the help of the use of the Min-Max principle. In our case, since the inclusion $V^N_{\omega,\beta,h} \subset V_{\omega,\beta}$ does not hold, our analysis must use more complex techniques, as the Anselone theory, that we have used in the case of the closed waveguides (cf. [17]). However, considering the results of [11] and these one obtained in the case of the closed waveguides, it is reasonable to enounce these two conjectures:

**Conjecture 1:** Under some standard assumptions of regularity of the functions $\varepsilon$ and $\mu$ and on the mesh $\tau_h$, we have the error estimate:

$$\left|\lambda_n(\omega,\beta) - \lambda_n^{h,N}(\omega,\beta)\right| \;\leq\; C_{s,n}(\omega,\beta)\,[h^2 + N^{-s}]$$

for every $s > 0$, the constant $C_{s,n}(\omega,\beta)$ depending regularly on $(\omega,\beta)$ when $0 < \omega < \beta c_\infty$.

**Conjecture 2:** If $\omega$ is a simple root of $\lambda_n(\omega,\beta) \;=\; \omega^2$, then for $N$ large enough and $h$ small enough, the equation $\lambda_n^{h,N}(\omega,\beta) \;=\; \omega^2$ admits a unique solution $\omega_{N,h}$ in the neighborhood of $\omega$. Moreover, one has the error estimate:

$$|\omega - \omega_{N,h}| \;\leq\; \mathrm{Cste}\,[h^2 + N^{-s}]$$

**Remark 5.5** *it is not surprising to see in the conjectures, a precision of special type with respect to the parameter $N$ (that is a convergence quicker than any power of $\frac{1}{N}$), because the approximation of the series is a spectral type approximation [3]. In fact, we can expect an exponential rate of convergence with respect to $N$.*

# 6 Numerical Results

We present in this section some numerical results, whose main purpose is the validation of the method. We study in this paragraph, a circular cylindrical waveguide of radius 1, whose permittivity, respectively the permeability is $\varepsilon_i = 2$, $\mu_i = 1$, in the core, whereas the characteristics of the cladding are $\varepsilon_\infty = 1.$ and $\mu_\infty = 1.$ (see figure 6.1). The interest of this example lies in the fact

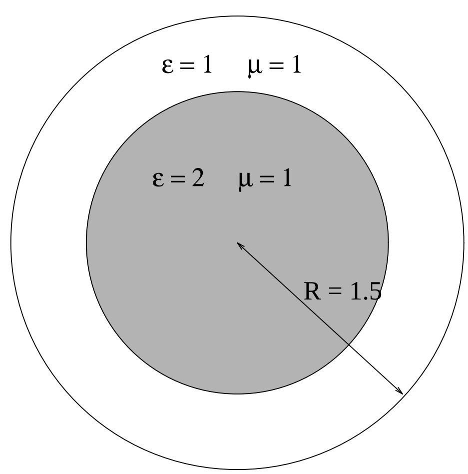



Figure 6.1: Circular Waveguide of Radius 1

that the guided modes can be computed analytically with the help of the separation of variables in polar coordinates $(r,\theta)$ (cf. D. Marcuse [21] or J.P. Pocholle [25]). The dispersion relations

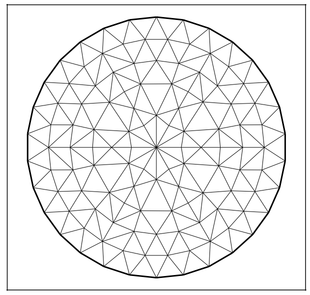


Figure 6.2: Mesh of the Circular Waveguide -$R = 1.5$

corresponding to modes in $e_r e^{in\theta}$ are:

$$(E_n) \quad n^2 \varepsilon_\infty \mu_\infty \gamma^2 (\frac{1}{\Delta} - 1)^2 \frac{\beta^4}{R^2 \alpha^4 \alpha_\Delta^4} = \left[\frac{\mu_i}{\alpha_\Delta} \frac{J_n'(\alpha_\Delta R)}{J_n(\alpha_\Delta R)} + \frac{\mu_\infty}{\alpha} \frac{K_n'(\alpha R)}{K_n(\alpha R)}\right] \times \left[\frac{\varepsilon_i}{\alpha_\Delta} \frac{J_n'(\alpha_\Delta R)}{J_n(\alpha_\Delta R)} + \frac{\varepsilon_\infty}{\alpha} \frac{K_n'(\alpha R)}{K_n(\alpha R)}\right],$$

$$\text{where} \quad \left\{ \begin{array}{lll} \Delta = \dfrac{\varepsilon_\infty \mu_\infty}{\varepsilon_i \mu_i} < 1 & \alpha = \beta\sqrt{1-\gamma^2} = \sqrt{\beta^2 - \omega^2 \varepsilon_\infty \mu_\infty} \\ \gamma^2 = \dfrac{\omega^2}{\beta^2} \varepsilon_\infty \mu_\infty \in ]\Delta, 1[ & \alpha_\Delta = \beta \sqrt{\dfrac{\gamma^2}{\Delta} - 1} \end{array} \right.$$

and where $J_n$ (respectively $K_n$) denotes the Bessel (respectively modified Bessel) function of first (respectively second) kind. For each $n \geq 0$, there exists an increasing sequence $\beta_m^n$ of thresholds such that, if $\beta_m^n < \beta < \beta_{m+1}^n$, the equation $(E_n)$ admits exactly $m$ solutions. These thresholds can be themselves defined by a transcendental equation (cf. R. Djellouli [10]). In order to obtain some reference solutions, we have solved numerically the $(E_n)$ equations. We give in the table (6.1), these "exact" solutions for some values of $\beta$.

| $\beta$ | 1 | 2.2 | 2.4049 | 2.6 |
|---|---|---|---|---|
| $N(\beta)$ | 2 | 2 | 4 | 4 |
| $\omega_1(\beta)^2$ | 0.993253 (n=1) | 3.762625 (n=1) | 4.321575 (n=1) | 4.883073 (n=1) |
| $\omega_2(\beta)^2$ | 0.993253 (n=1) | 3.762625 (n=1) | 4.321575 (n=1) | 4.883073 (n=1) |
| $\omega_3(\beta)^2$ | | | 5.783516 (n=0) | 6.526149 (n=0) |
| $\omega_4(\beta)^2$ | | | 5.783529 (n=0) | 6.619999 (n=0) |

Table 6.1: Reference Solutions of $E_n$ - $n$ harmonic in $\theta$

**1) Presentation of the calculus:** The computations have been achieved with a non structured mesh, consisting of 330 degrees of freedom, 210 triangles and 121 vertices (see figure 6.2). The results concerning the modes have been obtained with a refined mesh (1392 degrees of freedom). Let us notice that this mesh satisfies the geometrical assumption 5.1. We have set the artificial boundary at $R = 1.5$ and the order of the truncation $N$ has been fixed to 100. (For a discussion of the good choice of $N$ with respect to $h$, see for instance J.B. Keller and D. Givoli [18].)

**2) Results concerning the eigenvalues:** In figures 6.3, we represent the variations of the functions $\gamma \rightarrow \lambda_n(\beta c_\infty \gamma, \beta) - \beta^2 c_\infty^2 \gamma^2$, where $\gamma = \dfrac{\omega}{\beta c_\infty}$. One intersection point between one of these curves and the $x-$ axis reveals the existence of a guided mode. On figure 6.3 case $\beta = 1$, we observe the existence of one guided wave, which is double and associated to $n = 1$, whose phase velocity $\dfrac{\omega^2}{\beta^2}$ is closed to $c_\infty^2$. The vertical segment corresponds to the "exact" solution. The good coincidence between the numerical solution and the exact one illustrates the quality of our method.

As expected, when $\beta$ increases, the curves are translated "downstairs" and the number of solutions increases. For instance, we get four solutions for $\beta = 2.6$ (do not forget that the first solution is a double one) which still show a good agreement between the exact solutions and the approximate ones.

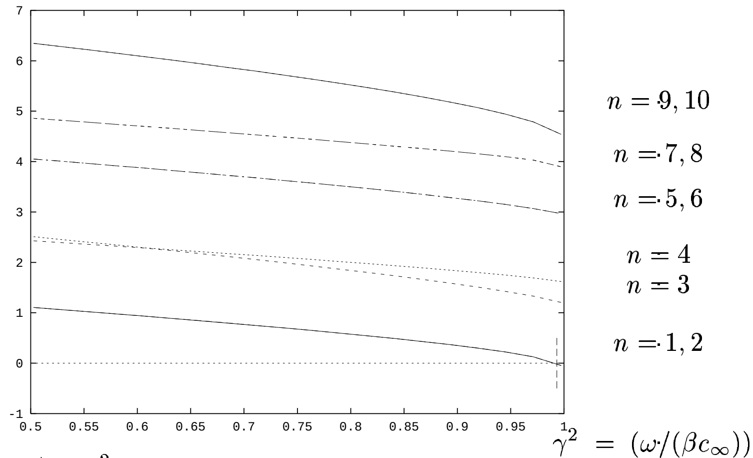
**3) Results concerning the guided modes:** In figures 6.4,6.5, 6.6, we have only represented the transverse components of the field. In each case, we first show the field in the computational domain, and then in a larger region. This second representation has the interest to illustrate the concentration of the transverse energy of the mode.

In figure 6.4, we have represented the first two modes (double mode) for $\beta = 2.6$ corresponding to the value $\omega = 2.2108$ (this value is the numerical one, the exact solution being $\omega = 2.2097$).

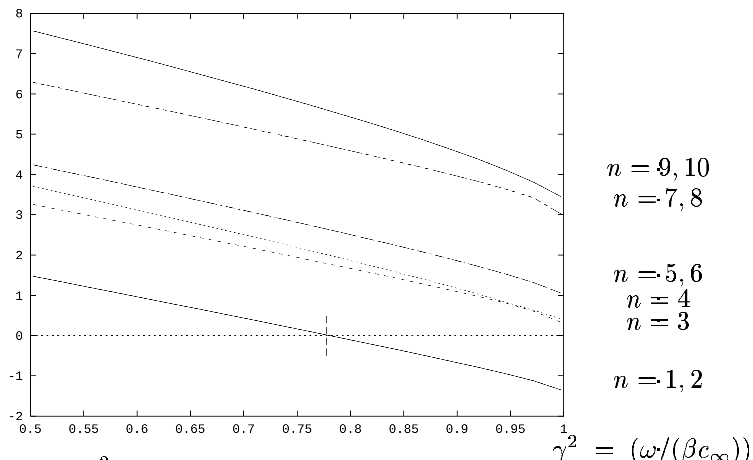
The case $\beta = 2.45$ is interesting because it is located just after the third and fourth thresholds. We observe on figure 6.5 that the third and fourth modes are not well confined, which was expected.

Finally, in order to represent an higher order mode, we have chosen the $\beta = 6$, which is greater than the $10^{th}$ threshold and represented the field corresponding to the $10^{th}$ mode. We remark the very good concentration of the mode in the neighborhood of the core, and the "vortices" characterizing the fact we look at a higher order mode.

*Case* : $\beta = 1$ $\lambda_n(\omega, \beta) - \omega^2$ $1 \leq n \leq 10$




*Case* : $\beta = 2.2$ $\lambda_n(\omega, \beta) - \omega^2$ $1 \leq n \leq 10$




*Case* : $\beta = 2.6$ $\lambda_n(\omega, \beta) - \omega^2$ $1 \leq n \leq 10$

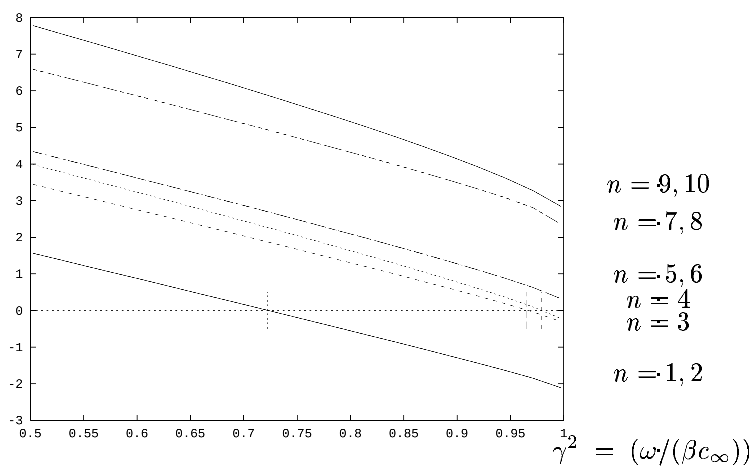



Figure 6.3: Circular Guide - $\beta = 1$, 2.2 or 2.6 - $N = 100$

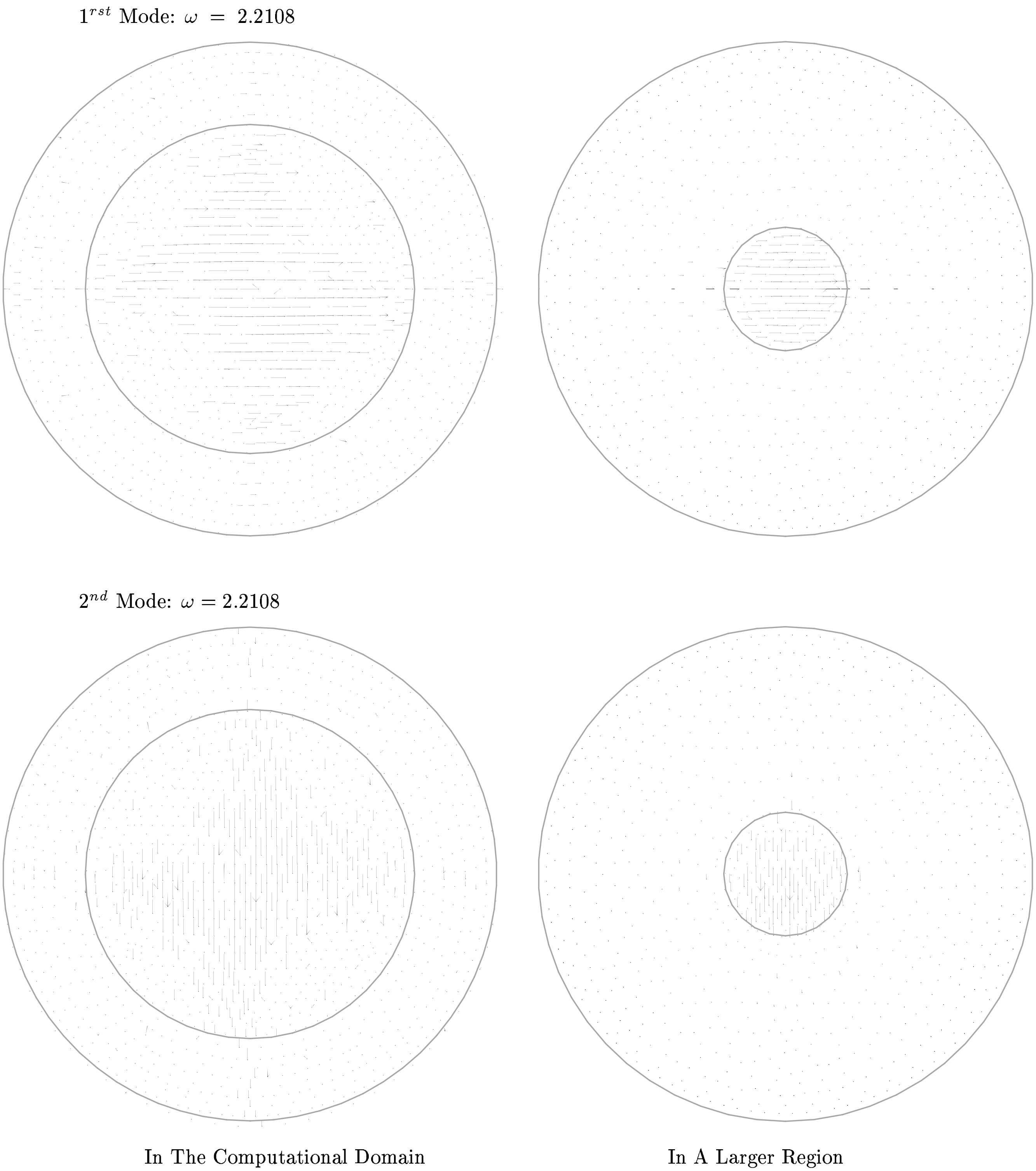
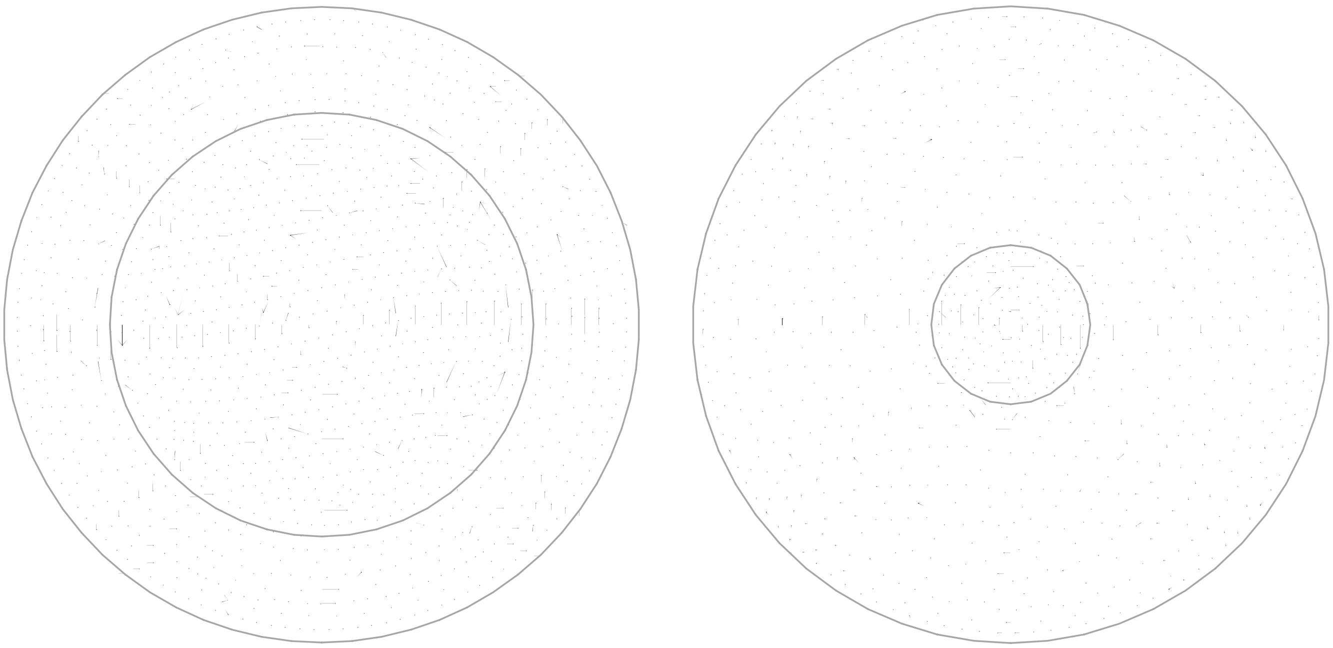
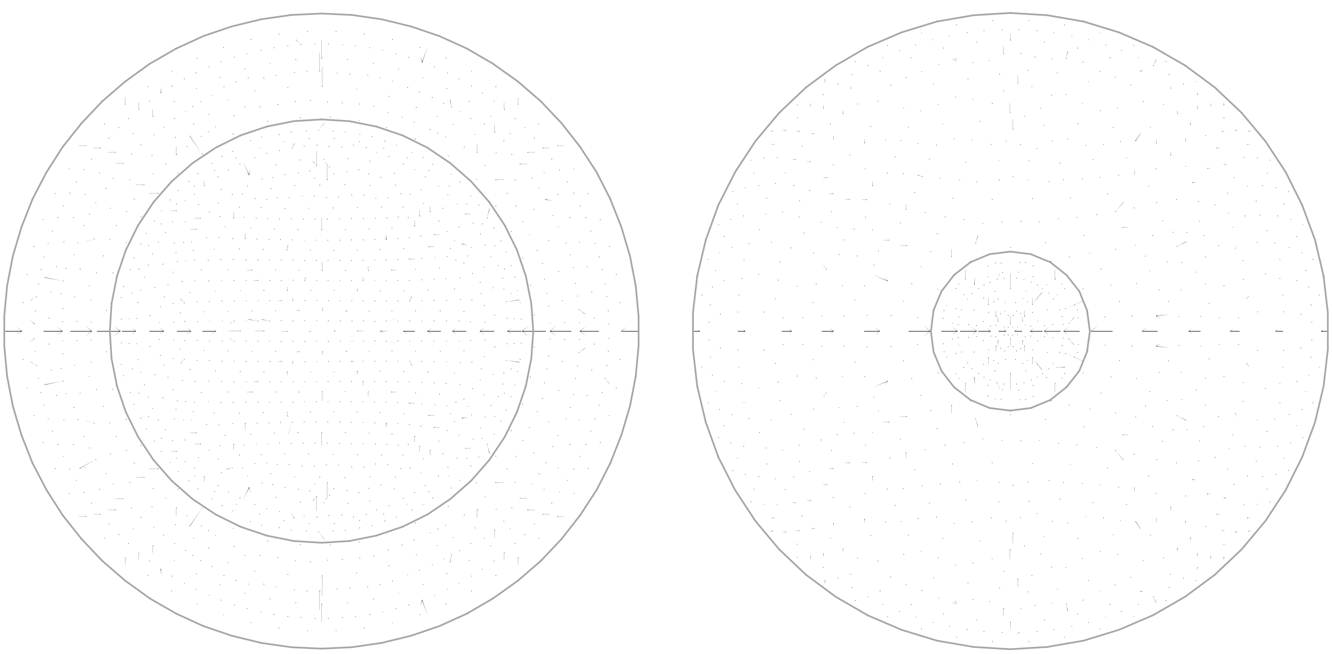



Figure 6.4: Circular Guide - $\beta = 2.6$ - $N = 100$ - Guided Modes 1 and 2

$3^{rd}$ Mode: $\omega \ = \ 2.4427$



$4^{th}$ Mode: $\omega = 2.4464$



In The Computational Domain In A Larger Region

Figure 6.5: Circular Guide - $\beta = 2.45$ - $N = 100$ - Guided Modes 3 and 4

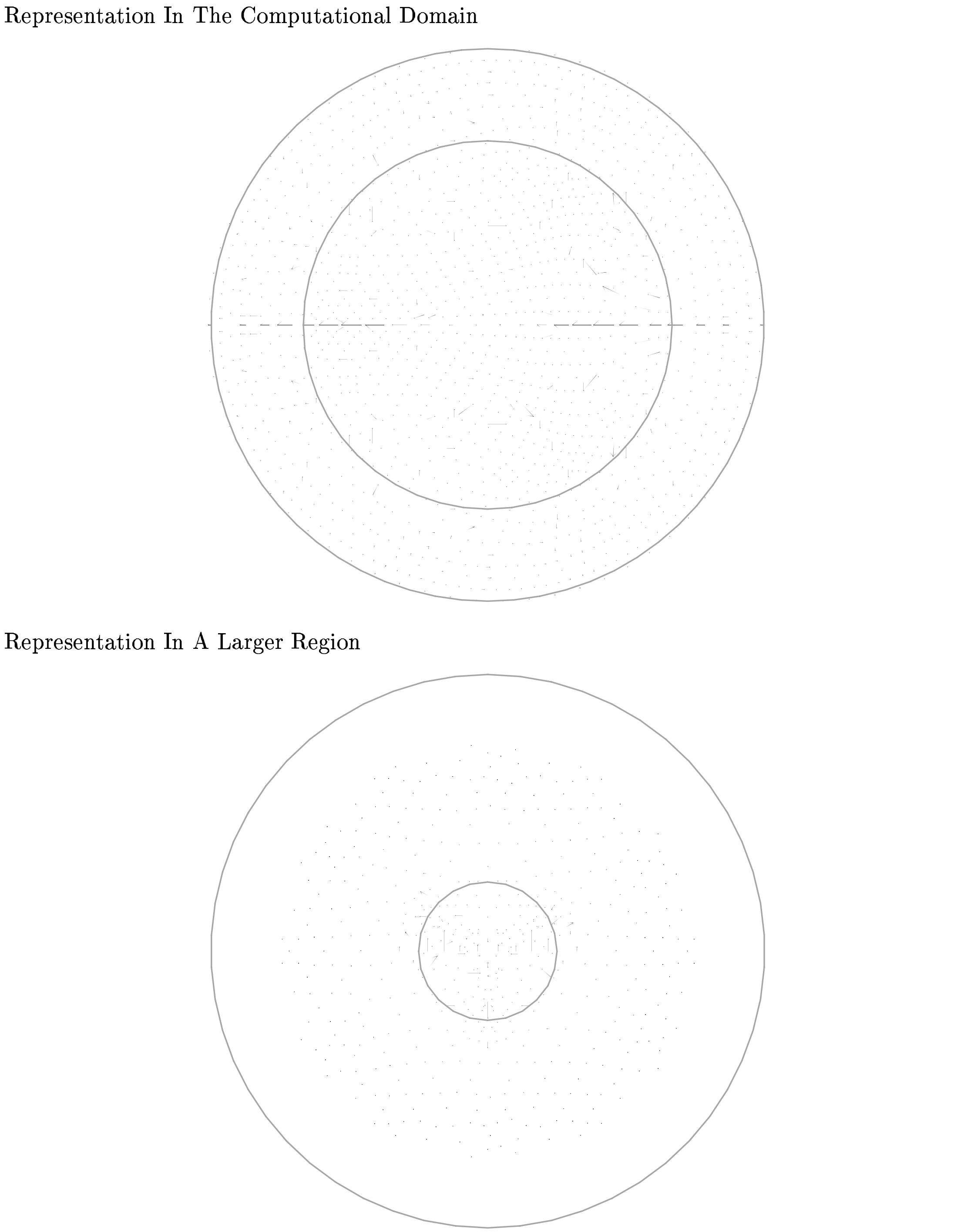



Figure 6.6: Circular Guide - $\beta = 6.$ - $\omega = 5.3073$ - $N = 100$ - $10^{th}$ Guided Mode

# A Construction of the operator $T_R(\omega,\beta)$

According to the definition of $T_R(\omega,\beta)$ (see definition 2.1), the first step of the construction of $T_R(\omega,\beta)$ consists in solving the exterior problem $(P^e_{\omega,\beta})$.

## A.1 Computation of the solution of the exterior problem:

Let $\phi$ be in $H^{-\frac{1}{2}}(\mathrm{rot}_\beta,\Gamma_R)$ and $(\omega,\beta)\in(\mathbb{R}^{\star+})^2$ satisfying the inequality $\beta^2-\dfrac{\omega^2}{c_\infty^2} > 0$, we recall that the problem $(P^e_{\omega,\beta})$ is written as follows:

$$(P^e_{\omega,\beta})\quad\left\{\begin{array}{l}\text{Find}\quad u_e\in H(\mathrm{rot}_\beta,\Omega_e)\\[2mm] \mathrm{rot}^\star_\beta(\mathrm{rot}_\beta u_e)=\dfrac{\omega^2}{c_\infty^2}u_e\\[2mm] \quad(A.1)\qquad (u_e\wedge n)\wedge n\;=\;\phi\;\text{ on }\;\Gamma_R.\end{array}\right.$$

Using the formula $\mathrm{rot}^\star_\beta(\mathrm{rot}_\beta u)\;=\;\nabla_\beta(\mathrm{div}_\beta u)-\Delta_\beta u$ and noting that the unique solution (cf. Lemma 2.2) of $(P^e_{\omega,\beta})$ satisfies necessarily $\mathrm{div}_\beta u\;=\;0$, we obtain the equivalences (we drop now the index $e$ for simplicity):

$$(P^\omega_e)\iff\left\{\begin{array}{l}-\Delta_\beta u\;=\;\dfrac{\omega^2}{c_\infty^2}u\\[2mm] \mathrm{div}_\beta u=0\\[2mm] (u\wedge n)\wedge n\;=\;\phi\;\text{ on }\;\Gamma_R\end{array}\right.\iff\left\{\begin{array}{l}u_3\;=\;\dfrac{1}{\beta}div\boldsymbol{u}\\[2mm] -\Delta\boldsymbol{u}_i+(\beta^2-\omega^2/c_\infty^2)\boldsymbol{u}_i\;=\;0\text{ for }\;i=1,2\\[2mm] (u\wedge n)\wedge n\;=\;\phi\;\text{ on }\;\Gamma_R.\end{array}\right.$$

In the sequel, we use polar coordinates $(x_1=r\cos\theta, x_2=r\sin\theta)$. The equations $-\Delta\boldsymbol{u}_i+(\beta^2-\dfrac{\omega^2}{c_\infty^2})\boldsymbol{u}_i=0$ for $i=1,2$ are then written:

$$\frac{\partial^2\boldsymbol{u}_i}{\partial r^2}+\frac{1}{r^2}\frac{\partial\boldsymbol{u}_i}{\partial r}+\frac{1}{r^2}\frac{\partial^2\boldsymbol{u}_i}{\partial\theta^2}-(\beta^2-\frac{\omega^2}{c_\infty^2})\boldsymbol{u}_i=0. \tag{A.2}$$

Using the decomposition in Fourier expansion of $\boldsymbol{u}_i$ $(\boldsymbol{u}_i(r,\theta)=\displaystyle\sum_{i=0}^{+\infty}\boldsymbol{u}_i^n(r)e^{in\theta})$, we obtain:

$$\frac{d^2\boldsymbol{u}_i^n}{dr^2}+\frac{1}{r}\frac{d\boldsymbol{u}_i^n}{dr}-(\frac{n^2}{r^2}+(\beta^2-\frac{\omega^2}{c_\infty^2}))\boldsymbol{u}_i^n=0. \tag{A.3}$$

Setting $\alpha:=\sqrt{\beta^2-\dfrac{\omega^2}{c_\infty^2}}$, the solutions of (A.3) are linear combinations of $K_n(\alpha r)$ and of $I_n(\alpha r)$ (cf. [1]). But due to the asymptotic behaviour of $K_n(z)$ and $I_n(z)$ when $z$ tends to $\infty$, only $L^2$ solutions of (A.3) are:

$$\boldsymbol{u}_1^n\;=\;A_n^1\,K_n(\alpha r)\quad,\quad\boldsymbol{u}_2^n\;=\;A_n^2\,K_n(\alpha r).$$

Then, it is useful to split up the vector field $u$ in cylindrical coordinates (for details, see (2.6) and figure 2.1):

$$u_r\;=\;u_1\cos\theta+u_2\sin\theta\quad;\quad u_\theta\;=\;-u_1\sin\theta+u_2\cos\theta,$$

from which we deduce that:

$$\begin{aligned}u_r\;&=\;\frac{1}{2}\sum_{n\in\mathbb{Z}}(A_n^1-iA_n^2)K_n(\alpha r)e^{i(n+1)\theta}+\frac{1}{2}\sum_{n\in\mathbb{Z}}(A_n^1+iA_n^2)K_n(\alpha r)e^{i(n-1)\theta}\\ &=\;\sum_{n\in\mathbb{Z}}\left(C_{n-1}K_{n-1}(\alpha r)+D_{n+1}K_{n+1}(\alpha r)\right)e^{in\theta}\end{aligned}$$

where

$$C_n \;=\; \frac{1}{2}(A_n^1 - iA_n^2) \quad \text{and} \quad D_n \;=\; \frac{1}{2}(A_n^1 + iA_n^2).$$

In the same way, we obtain for $u_\theta$ the following relation:

$$u_\theta \;=\; i\sum_{n\in\mathbb{Z}} \left(C_{n-1}K_{n-1}(\alpha r) - D_{n+1}K_{n+1}(\alpha r)\right) e^{in\theta}. \tag{A.4}$$

For convenience, we will work with a new expression of $u_r$ and $u_\theta$. Indeed, the relations (5.23) (see also [1]) allow us to express $u_r$ and $u_\theta$ with respect to $K_n(\alpha r)$ and $K_n'(\alpha r)$:

$$u_r \;=\; \sum_{n\in\mathbb{Z}} \left\{-(C_{n-1}+D_{n+1})K_n'(\alpha r) - (C_{n-1}-D_{n+1})\frac{n}{\alpha r}K_n(\alpha r)\right\} \; e^{in\theta} \tag{A.5}$$

$$u_\theta \;=\; -i\sum_{n\in\mathbb{Z}} \left\{(C_{n-1}-D_{n+1})K_n'(\alpha r) + (C_{n-1}+D_{n+1})\frac{n}{\alpha r}K_n(\alpha r)\right\} \; e^{in\theta}. \tag{A.6}$$

Let us now compute the third component of $u$:

$$\beta u_3 \;=\; \mathrm{div}\boldsymbol{u} \;=\; \frac{1}{r}\left(\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_\theta}{\partial\theta}\right) \;=\; \frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}, \tag{A.7}$$

which gives in Fourier expansion:

$$\begin{aligned}
u_3^n \;&= \beta^{-1} \;\; \{\, \alpha(C_{n-1}K'_{n-1}(\alpha r) + D_{n+1}K'_{n+1}(\alpha r) + \frac{1}{r}(C_{n-1}K_{n-1}(\alpha r) + D_{n+1}K_{n+1}(\alpha r)) \\
&\qquad + \frac{i^2 n}{r}(C_{n-1}K_{n-1}(\alpha r) - D_{n+1}K_{n+1}(\alpha r)) \,\} \\
&= \beta^{-1} \;\; \{\, \frac{C_{n-1}}{r}((\alpha r)K'_{n-1}(\alpha r) - (n-1)K_{n-1}(\alpha r)) \\
&\qquad + \frac{D_{n+1}}{r}((\alpha r)K'_{n+1}(\alpha r) + (n+1)K_{n+1}(\alpha r)) \,\},
\end{aligned}$$

that is to say, thanks to the relations (5.23):

$$u_3 \;=\; \left(-\frac{\alpha}{\beta}\right) \sum_{n\in\mathbb{Z}} (C_{n-1}+D_{n+1})K_n(\alpha r)\; e^{in\theta}. \tag{A.8}$$

From the formulae (A.5), (A.6) and (A.8), it remains to define the constants $C_n$ and $D_n$, with the help of the boundary conditions (A.1). If $\phi \;=\; -\phi_\theta\,\vec{e_\theta} - \phi_3\,\vec{e_3}$ where $(\phi_\theta,\phi_3) \in H^{-\frac{1}{2}}(\Gamma_R) \times H^{\frac{1}{2}}(\Gamma_R)$, the boundary condition (A.1) gives:

$$u_\theta(R,\theta) \;=\; \phi_\theta(\theta) \qquad u_3(R,\theta) \;=\; \phi_3(\theta) \qquad \text{for} \;\; \theta\in[0,2\pi]. \tag{A.9}$$

Therefore, using Fourier expansion of $\phi_3$ and $\phi_\theta$:

$$u_{3/\Gamma_R} = \phi_3 \quad\Longleftrightarrow\quad \phi_3^n \;=\; -\frac{\alpha}{\beta}\,(C_{n-1}+D_{n+1})K_n(\alpha R) \quad \forall n\in\mathbb{Z} \tag{A.10}$$

and

$$u_{\theta/\Gamma_R} \;=\; \phi_\theta \iff \forall n\in\mathbb{Z} \quad \left\{\begin{array}{l} \phi_\theta^n \;=\; -i\;\{(C_{n-1}-D_{n+1})\;K_n'(\alpha R) \\[2mm] +\;(C_{n-1}+D_{n+1})\dfrac{n}{\alpha R}\;K_n(\alpha R)\;\}. \end{array}\right. \tag{A.11}$$

We deduce from (A.10) and (A.11) that

$$\left\{\begin{array}{l} C_{n-1}+D_{n+1} = -\dfrac{\beta}{\alpha}\dfrac{\phi_3^n}{K_n(\alpha R)} \\[3mm] C_{n-1}-D_{n+1} = \dfrac{i\phi_\theta^n + \frac{\beta n}{\alpha^2 R}\phi_3^n}{K_n'(\alpha R)} \end{array}\right. \qquad \forall n\in\mathbb{Z}, \tag{A.12}$$

which gives the constants $C_{n-1}$ and $D_{n+1}$ (cf. A.12), and finally:

$$\left\{\begin{array}{ll}(i) & u_r = \sum_{n\in\mathbb{Z}}\{\ \frac{\beta}{\alpha}\phi_3^n\frac{K_n'(\alpha r)}{K_n(\alpha R)} - (i\phi_\theta^n + \frac{\beta n}{\alpha^2 R}\phi_3^n)\frac{n}{\alpha r}\frac{K_n(\alpha r)}{K_n'(\alpha R}\ \}e^{in\theta} \\ (ii) & u_\theta = \sum_{n\in\mathbb{Z}}\{\ (\phi_\theta^n - i\frac{\beta n}{\alpha^2 R}\phi_3^n)\frac{K_n'(\alpha r)}{K_n'(\alpha R)} - i\frac{\beta n}{\alpha^2 r}\phi_3^n\frac{K_n(\alpha r)}{K_n(\alpha R)}\ \}e^{in\theta} \\ (iii) & u_3 = \sum_{n\in\mathbb{Z}}\phi_3^n\frac{K_n(\alpha r)}{K_n(\alpha R)}e^{in\theta}\end{array}\right. \tag{A.13}$$

## A.2 Computation of $\mathrm{rot}_\beta u$ in cylindrical coordinates:

In fact, we want to compute, from $u$, the tangential trace of $\mathrm{rot}_\beta u$ on $\Gamma_R$. More precisely, we want to compute $\mathrm{rot}_\beta u \wedge e_r$ for $r = R$:

$$\mathrm{rot}_\beta u \wedge e_{r/\Gamma_R} = (\mathrm{rot}_\beta u)_3\vec{e_\theta} - (\mathrm{rot}_\beta u)_\theta\vec{e_3}\ _{/\ (r=R)}. \tag{A.14}$$

From the expression of $\mathrm{rot}_\beta u$ in cylindrical coordinates, we get:

$$\left\{\begin{array}{lclcl}(\mathrm{rot}_\beta u)_3 & = & \frac{\partial u_\theta}{\partial r} + \frac{1}{r}u_\theta - \frac{1}{r}\frac{\partial u_r}{\partial\theta} & = & \sum_{n\in\mathbb{Z}}(\frac{du_\theta^n}{dr}(r) + \frac{1}{r}u_\theta^n(r) - \frac{in}{r}u_r^n)e^{in\theta} \\ (\mathrm{rot}_\beta u)_\theta & = & (\beta u_r - \frac{\partial u_3}{\partial r}) & = & \sum_{n\in\mathbb{Z}}(\beta u_r^n(r) - \frac{du_3^n}{dr}(r))e^{in\theta}.\end{array}\right. \tag{A.15}$$

Therefore, we have, still using (5.23):

$$\begin{array}{rcl}(\mathrm{rot}_\beta u)_3^n & = & i\alpha C_{n-1}K_{n-1}'(\alpha r) - i\alpha D_{n+1}K_{n+1}'(\alpha r) + \frac{i}{r}(C_{n-1}K_{n-1}(\alpha r) - D_{n+1}K_{n+1}(\alpha r)) \\ & & -\frac{in}{r}(C_{n-1}K_{n-1}(\alpha r) + D_{n+1}K_{n+1}(\alpha r)) \\ & = & (-\alpha i)(C_{n-1} - D_{n+1})K_n(\alpha r),\end{array}$$

that is to say (see (A.12)):

$$(\mathrm{rot}_\beta u)_3 \ = \ \sum_{n\in\mathbb{Z}}\{\alpha\phi_\theta^n - i\frac{\beta n}{\alpha R}\phi_3^n\}\frac{K_n(\alpha r)}{K_n'(\alpha R)}e^{in\theta}. \tag{A.16}$$

In the same way, from (A.15) and (A.13 (i),(iii)) we obtain:

$$(\mathrm{rot}_\beta u)_\theta^n \ = \ \frac{\beta^2}{\alpha}\phi_3^n\frac{K_n'(\alpha r)}{K_n(\alpha R)} - (\frac{\beta^2 n}{\alpha^2 R}\phi_3^n + i\beta\phi_\theta^n)\frac{n}{\alpha r}\frac{K_n(\alpha r)}{K_n'(\alpha R)} - \alpha\phi_3^n\frac{K_n'(\alpha r)}{K_n(\alpha R)}, \tag{A.17}$$

that we can write:

$$(\mathrm{rot}_\beta u)_\theta = \sum_{n\in\mathbb{Z}}\{\ \frac{\beta^2-\alpha^2}{\alpha}\phi_3^n\frac{K_n'(\alpha r)}{K_n(\alpha R)} - (\frac{\beta^2 n}{\alpha^2 R}\phi_3^n + i\beta\phi_\theta^n)\frac{n}{\alpha r}\frac{K_n(\alpha r)}{K_n'(\alpha R)}\}e^{in\theta}. \tag{A.18}$$

We can now express the operator $T_R(\omega,\beta)$ with the help of a countable set of 2 by 2 matrices, using formulae (A.16) and (A.18). We recall that, by definition of $T_R(\omega,\beta)$, $u$ being the solution of $(P_e^\omega)$ with $(u\wedge n)\wedge n_{/\Gamma_R} \ = \ \phi$, we have

$$T_R(\omega,\beta)(\phi) \ = \ \mathrm{rot}_\beta u\wedge n_{/\Gamma_R} \ = \ (\mathrm{rot}_\beta u)_3\vec{e_\theta} - (\mathrm{rot}_\beta u)_\theta\vec{e_3}_{/\Gamma_R}$$

Finally, we find the expression of $T_R(\omega,\beta)$, given in Theorem 2.1.

# B Appendix: Proof of Theorem 3.2 (The density of $V_{\omega,\beta}(\Omega_i)$ in $H_\beta(\Omega_i)$)

**Proof** We divide the proof into two parts:
**(i)** we construct an operator $B_{\omega,\beta}$, whose domain $\mathcal{D}(B_{\omega,\beta})$ is dense in $H(\mathrm{rot}_\beta,\Omega_i)$ and such a way that

$$\forall u \in \mathcal{D}(B_{\omega,\beta}) \quad \varepsilon^{-1}\mathrm{rot}^\star_\beta u \;\in\; V_{\omega,\beta}(\Omega_i).$$

**(ii)** Using this property, we prove directly the density of $V_{\omega,\beta}(\Omega_i)$ in $H_\beta(\Omega_i)$.

**(i)** - Let us introduce the unbounded operator $B_{\omega,\beta}$ of $L^2(\Omega_i)^3$, equipped of the norm $\| \, \|_{0,\Omega_i}$, defined by

$$\left\{\begin{array}{l} \mathcal{D}(B_{\omega,\beta}) = \quad \{ \qquad u_i \in H(\mathrm{rot}_\beta,\Omega_i) \; ; \; \mathrm{rot}_\beta(\varepsilon^{-1}\mathrm{rot}^\star_\beta u_i) \in L^2(\Omega_i)^3; \\ \qquad\qquad\qquad (\mathrm{rot}^\star_\beta u_i \wedge n)\wedge n_{/\Gamma_R} = S_R^{inv}(\omega,\beta)(\mathrm{rot}^\star_{\boldsymbol{\beta}}\boldsymbol{u_i}\cdot n_{/\Gamma_R}) \quad \} \\ B_{\omega,\beta}u_i \;=\; \mathrm{rot}_\beta(\varepsilon^{-1}\mathrm{rot}^\star_\beta u_i) \quad \forall u_i \in \mathcal{D}(B_{\omega,\beta}). \end{array}\right. \tag{B.1}$$

We now prove that $B_{\omega,\beta}$ is self-adjoint so that its domain is dense in $H(\mathrm{rot}_\beta,\Omega_i)$.
Thanks to Green's formula (1.14), we have

$$\begin{aligned} (B_{\omega,\beta}u_i, v_i)_{0,\Omega_i} &= (\mathrm{rot}_\beta(\varepsilon^{-1}\mathrm{rot}^\star_\beta u_i), v_i)_{0,\Omega_i} \\ &= (\varepsilon^{-1}\mathrm{rot}^\star_\beta u_i, \mathrm{rot}^\star_\beta v_i)_{0,\Omega_i} - < (\varepsilon^{-1}\mathrm{rot}^\star_\beta u_i \wedge n)\wedge n, v_i\wedge n >_{/\Gamma_R} \\ &= (\varepsilon^{-1}\mathrm{rot}^\star_\beta u_i, \mathrm{rot}^\star_\beta v_i)_{0,\Omega_i} - \varepsilon_\infty^{-1} < S_R^{inv}(\omega,\beta)(\mathrm{rot}^\star_{\boldsymbol{\beta}}\boldsymbol{u_i}\cdot n), v_i\wedge n >_{/\Gamma_R} . \end{aligned}$$

We then define the bilinear form $b_{\omega,\beta}$ associated to $B_{\omega,\beta}$, as follows:

$$\left\{\begin{array}{l} \mathcal{D}(b_{\omega,\beta}) \;=\; H(\mathrm{rot}_\beta,\Omega_i) \text{ which is dense to } L^2(\Omega_i)^3 \\ b_{\omega,\beta}(u,v) = (\varepsilon^{-1}\mathrm{rot}^\star_\beta u, \mathrm{rot}^\star_\beta v)_{0,\Omega_i} - \varepsilon_\infty^{-1} < S_R^{inv}(\omega,\beta)(\mathrm{rot}^\star_{\boldsymbol{\beta}}\boldsymbol{u}\cdot n), v\wedge n >_{/\Gamma_R} . \end{array}\right. \tag{B.2}$$

**Remark B.1** *To see that definitions (B.1) and (B.2) make sense, it is useful to recall that* $S_R^{inv}(\omega,\beta)$ *belongs to* $\mathcal{L}(H^{-\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(rot_\beta,\Gamma_R))$. *Since* $rot^\star_\beta u \in H(div_\beta,\Omega_i)$ *then* $rot^\star_\beta u\cdot\, n_{/\Gamma_R} \in H^{-\frac{1}{2}}(\Gamma_R)$. *These two facts justify (B.1) and (B.2). Moreover, by duality between* $H^{-\frac{1}{2}}(rot_\beta,\Gamma_R)$ *and* $H^{-\frac{1}{2}}(div_\beta,\Gamma_R)$, *and continuity of* $S_R^{inv}(\omega,\beta)$, *we have, using trace theorems:*

$$\begin{aligned} | < S_R^{inv}(\omega,\beta)(rot^\star_{\boldsymbol{\beta}}\boldsymbol{u}\cdot n), v\wedge n >_{/\Gamma_R} | &\leq C(\omega,\beta)\|rot^\star_\beta u\cdot n\|_{H^{-\frac{1}{2}}(\Gamma_R)}\|v\wedge n\|_{H^{-\frac{1}{2}}(div_\beta,\Gamma_R)} \\ &\leq C_1(\omega,\beta)\|rot^\star_\beta u\|_{H(div_\beta,\Omega_i)}\|v\|_{H(rot_\beta,\Omega_i)} , \end{aligned}$$

*which proves the continuity of the bilinear form* $b_{\omega,\beta}$, *since* $\|rot^\star_\beta u\|_{H(div_\beta,\Omega_i)} = \|rot^\star_\beta u\|_{L^2(\Omega_i)}$.

Let us show that $b_{\omega,\beta}$ is symmetric, which is not obvious from (B.2). In polar coordinates, we have

$$\left\{\begin{array}{l} v\wedge n_{/\Gamma_R} \;=\; (v_3(\mathrm{R},\theta), -v_\theta(\mathrm{R},\theta)) \;=\; \displaystyle\sum_{n\in\mathbb{Z}}(v_3^n, -v_\theta^n)\, e^{in\theta} \\ \mathrm{rot}^\star_{\boldsymbol{\beta}}\boldsymbol{u}\cdot n_{/\Gamma_R} \;=\; (\mathrm{rot}^\star_\beta u)_r(\mathrm{R},\theta) \;=\; \dfrac{1}{\mathrm{R}}\dfrac{\partial u_3}{\partial\theta} + \beta u_\theta \;=\; \displaystyle\sum_{n\in\mathbb{Z}}(\frac{in}{R}u_3^n + \beta u_\theta^n)\, e^{in\theta}. \end{array}\right.$$

which gives:

$$- < S_R^{inv}(\omega,\beta)(\mathrm{rot}_{\boldsymbol{\beta}}^{\star}\boldsymbol{u}\cdot n), v\wedge n >_{/\Gamma_R} = \sum_{n\in\mathbb{Z}} \left( -\frac{in}{R}u_3^n + \beta(-u_\theta^n) \right) \begin{pmatrix} i\alpha^n \\ \gamma^n \end{pmatrix} (\overline{v_3^n}, -\overline{v_\theta^n})$$

$$= \sum_{n\in\mathbb{Z}} [M^n] \begin{pmatrix} u_3^n \\ -u_\theta^n \end{pmatrix} \cdot \begin{pmatrix} \overline{v_3^n} \\ -\overline{v_\theta^n} \end{pmatrix}$$

where

$$[M^n] = \begin{bmatrix} \frac{n}{R}\alpha^n & i\beta\alpha^n \\ -\frac{in}{R}\gamma^n & \beta\gamma^n \end{bmatrix}. \tag{B.3}$$

In fact, we have chosen the terms $\alpha^n$ and $\gamma^n$, in such a way that the matrix $M^n$ is hermitian, i.e. $-\frac{in}{R}\gamma^n = \overline{i\beta\alpha^n}$ or in other words $\alpha^n = \frac{n}{\beta R}\gamma^n$, which implies that the form $b_{\omega,\beta}$ is symmetric. Moreover the equality:

$$- < S_R^{inv}(\omega,\beta)(\mathrm{rot}_{\boldsymbol{\beta}}^{\star}\boldsymbol{u}\cdot n), v\wedge n >_{/\Gamma_R} = \sum_{n\in\mathbb{Z}} \frac{\gamma^n}{\beta}\left(\frac{in}{R}u_3^n + \beta u_\theta^n\right)\overline{\left(\frac{in}{R}v_3^n + \beta v_\theta^n\right)} \tag{B.4}$$

shows the positivity of $b_{\omega,\beta}$ since $\gamma^n > 0 \quad \forall n \in \mathbb{Z}$ (see Lemma 3.2). Therefore, there exists $\alpha > 0$ such that

$$b_{\omega,\beta}(u,u) + \|u\|_{0,\Omega_i}^2 \geq \alpha\, \|u\|_{H(\mathrm{rot}_\beta,\Omega_i)}^2 \tag{B.5}$$

and consequently, the operator $B_{\omega,\beta}$ is selfadjoint (cf. [12]). As a consequence, by a classical theorem about selfadjoint operators, $\mathcal{D}(B_{\omega,\beta})$ is dense in $L^2(\Omega_i)^3$ and in $H(\mathrm{rot}_\beta,\Omega_i)$. Finally, we check that for any $v\in\mathcal{D}(B_{\omega,\beta})$, $u=\varepsilon^{-1}\mathrm{rot}_\beta^\star v$ belongs to $V_\beta(\Omega_i)$. Indeed, we have $\mathrm{div}_\beta(\varepsilon u)=0$ and by definition of $\mathcal{D}(B_{\omega,\beta})$, $\mathrm{rot}_\beta u\in L^2(\Omega_i)^3$. It remains to prove that $u\cdot n_{/\Gamma_R} = S_R(\omega,\beta)((u\wedge n)\wedge n)$. But since $v$ belongs to $\mathcal{D}(B_{\omega,\beta})$, we have $(u\wedge n)\wedge n) = S_R^{inv}(\omega,\beta)(u\cdot n_{/\Gamma_R})$, which yields, by Lemma (3.2), that $u\cdot n_{/\Gamma_R} = S_R(\omega,\beta)((u\wedge n)\wedge n)$.

**(ii)** - We can now prove the density of $V_{\omega,\beta}(\Omega_i)$ in $H_\beta(\Omega_i)$.
Let be $u\in H_\beta(\Omega_i)$, from Theorem 1.3,

$$\exists v\in H(\mathrm{rot}_\beta,\Omega_i) \text{ such that } u=\frac{1}{\varepsilon}\mathrm{rot}_\beta^\star v.$$

But, since $\mathcal{D}(B_{\omega,\beta})$ is dense in $H(\mathrm{rot}_\beta,\Omega_i)$,

$$\exists v^n\in\mathcal{D}(B_{\omega,\beta}) \text{ such that } v^n\longrightarrow v \text{ in } H(\mathrm{rot}_\beta,\Omega_i),$$

that is, particularly

$$\varepsilon^{-1}\mathrm{rot}_\beta^\star v^n \longrightarrow u \text{ in } L^2(\Omega_i)^3,$$

which completes the proof, since $\varepsilon^{-1}\mathrm{rot}_\beta^\star v^n \in V_{\omega,\beta}(\Omega_i)$ (see part (i)).

♦

# C Appendix: Proof of Theorem 4.1 (A generalized Helmholtz decomposition of $H(\mathbf{rot}_\beta,\Omega_i)$)

For the proof, we need to introduce a new boundary operator:

**Definition C.1** *We denote by $T_n$, the operator from $H^{\frac{1}{2}}(\Gamma_R)$ into $H^{-\frac{1}{2}}(\Gamma_R)$ as follows: (This operator is often entitled Dirichlet-Neumann or Steklov-Poincaré operator.) let be $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$, we have*

$$T_n(\varphi) \;=\; \frac{\partial \phi_i}{\partial n}_{/\Gamma_R},$$

*where $\phi_i \in H^1(\Omega_i)$ is the unique solution of*

$$div_\beta(\varepsilon \nabla_\beta \phi_i) \;=\; 0 \quad and \quad \phi_{i/\Gamma_R} \;=\; \varphi. \tag{C.1}$$

The operator $T_n$ has the following well-known properties:

**Lemma C.1** *$T_n$ is continuous, positive definite and symmetric from $H^{\frac{1}{2}}(\Gamma_R)$ into $H^{-\frac{1}{2}}(\Gamma_R)$.*

**Proof** See [26]. ♦

**Proof of Theorem 4.1**
We recall the aim of this proof. Let $u_i$ be given in $V_\beta(\Omega_i)$, we want to construct $(v_i, \phi) \in V_{\omega,\beta}(\Omega_i) \times H^1(\mathbb{R}^2)$ such a way that

$$u_i \;=\; v_i \;+\; \nabla_\beta \phi_{/\Omega_i}. \tag{C.2}$$

What must verify $v_i$ and $\phi$ ?

- $u_i \in H_\beta(\Omega_i)$ and $v_i \in H_\beta(\Omega_i) \quad \Rightarrow \quad \mathrm{div}_\beta(\varepsilon \nabla_\beta \phi) = 0$ in $\Omega_i$.
- $v_i$ must satisfy $\quad v_i \cdot n_{/\Gamma_R} \;=\; S_R(\omega,\beta)[(v_i \wedge n) \wedge n_{/\Gamma_R}]$. In other words, if $v_e$ is the solution of $(P^e_{\omega,\beta})$ associated to the boundary condition $(v_e \wedge n) \wedge n_{/\Gamma_R} = (v_i \wedge n) \wedge n_{/\Gamma_R} (= ((u_i - \nabla_\beta \phi_{/\Omega_i}) \wedge n) \wedge n_{/\Gamma_R})$, by definition of $S_R(\omega,\beta)$, we have:

  $$v_e \cdot n_{/\Gamma_R} \;=\; v_i \cdot n_{/\Gamma_R},$$

  that is also

  $$(u_i - \nabla_\beta \phi_{/\Omega_i}) \cdot n_{/\Gamma_R} \;=\; v_e \cdot n_{/\Gamma_R}.$$

The idea is first to construct $v_i$ and $\phi$ satisfying these properties. We shall see that we will be able to verify "a posteriori" that (C.2) holds. Our method will consist in constructing an extension $u_e \in V_\beta(\Omega_e)$ of $u_i$ and $\phi$ in $H^1(\mathbb{R}^2)$, from which we will define $v_i$. More precisely $u_e$ and $\phi$ will satisfy:

$$\left\{\begin{array}{ll} (i) & \mathrm{div}_\beta(\varepsilon \nabla_\beta \phi) = 0 \quad \text{in } \Omega_i \\[2ex] (ii) & v_e := (u_e - \nabla_\beta \phi) \text{ is the solution of the exterior problem } (P^e_{\omega,\beta}) \text{ associated} \\[2ex] & \text{to the boundary condition } (v_e \wedge n) \wedge n_{/\Gamma_R} = ((u_i - \nabla_\beta \phi_{/\Omega_i}) \wedge n) \wedge n_{/\Gamma_R}. \\[2ex] (iii) & v_e \cdot n_{/\Gamma_R} = (\boldsymbol{u_i} - \nabla \phi_{/\Omega_i}) \cdot n_{/\Gamma_R}. \end{array}\right. \tag{C.3}$$

Now, if $(u_e, \phi) \in V_\beta(\Omega_e) \times H^1(\mathbb{R}^2)$ satisfies (C.3), then the pair $(v_i, \phi)$ where $v_i$ is defined by $v_i \;=\; u_i \;-\; \nabla_\beta \phi_{/\Omega_i}$ will satisfy the decomposition (C.2). For this, we only have to show that $v_i$ belongs to $V_{\omega,\beta}(\Omega_i)$. Indeed:

(1) From (C.3) (i) and the fact that $u_i \in V_\beta(\Omega_i)$, we have $v_i$ belongs to $V_\beta(\Omega_i)$

(2) From the definition of the exterior problem $(P^e_{\omega,\beta})$, and of the trace operator $S_R(\omega,\beta)$, we deduce from $(ii)$ that

$$v_e \cdot n_{/\Gamma_R} \;=\; S_R(\omega,\beta)[(v_i \wedge n) \wedge n_{/\Gamma_R}], \tag{C.4}$$

then adding the point (C.3) (iii), we can conclude that

$$v_i \cdot n_{/\Gamma_R} \;=\; S_R(\omega,\beta)[(v_i \wedge n) \wedge n_{/\Gamma_R}].$$

From these two points, we conclude that $v_i \in V_{\omega,\beta}(\Omega_i)$.

We will see now, that satisfying (i),(ii) and (iii) of (C.3) leads us to define the pair $(u_e, \phi)$ as the solution of a variational problem. Indeed, setting $\phi_i = \phi_{/\Omega_i}$ and $\phi_e = \phi_{/\Omega_e}$, we note that $(u_e, \phi)$ will satisfy the following requirements:

- As we seek $u_e$ in $V_\beta(\Omega_e)$, from (C.3 (i),(ii)), we remark immediately that necessarily,

$$\begin{cases} \operatorname{div}_\beta(\varepsilon \nabla_\beta \phi_i) = 0 & \text{in } \Omega_i \\ \operatorname{div}_\beta(\varepsilon \nabla_\beta \phi_e) = 0 & \text{in } \Omega_e \quad (\operatorname{div}_\beta(\varepsilon v_e) = 0) \\ \phi_i = \phi_e & \text{in } \Gamma_R \quad (\phi \in H^1(\mathbb{R}^2)), \end{cases} \tag{C.5}$$

  whose consequence is the fact, that $\phi_i$ is completely defined by $\phi_e$. This is why in the sequel, we shall only seek to define $u_e$ and $\phi_e$.

- The problem (C.3 (ii)) can be rewritten as follows:

$$\begin{cases} \operatorname{rot}^\star_\beta(\operatorname{rot}_\beta(u_e - \nabla_\beta \phi_e)) = \dfrac{\omega^2}{c_\infty^2}(u_e - \nabla_\beta \phi_e) & \text{in } \Omega_e \\ ((u_e - \nabla_\beta \phi_e) \wedge n) \wedge n = ((u_i - \nabla_\beta \phi_i) \wedge n) \wedge n & \text{in } \Gamma_R. \end{cases} \tag{C.6}$$

  Due to the fact that $\nabla_\beta \phi_i \wedge n_{/\Gamma_R} = \nabla_\beta \phi_e \wedge n_{/\Gamma_R}$, and $\operatorname{rot}_\beta(\nabla_\beta \phi_e) = 0$, $(u_e, \phi_e)$ must satisfy:

$$\begin{cases} \operatorname{rot}^\star_\beta(\operatorname{rot}_\beta u_e) = \dfrac{\omega^2}{c_\infty^2}(u_e - \nabla_\beta \phi_e) & \text{dans } \Omega_e \\ (u_e \wedge n) \wedge n = (u_i \wedge n) \wedge n & \text{sur } \Gamma_R. \end{cases} \tag{C.7}$$

- It remains now to express the boundary condition (C.3(iii)) with respect to $u_e$ and $\phi_e$, that we can do with the help of the operator $T_n$. Indeed, from (C.5 (i)), we have

$$\frac{\partial \phi_i}{\partial n}/\Gamma_R = T_n(\phi_{e/\Gamma_R}), \tag{C.8}$$

  which allows us to rewrite the boundary condition (C.3(iii)) as follows:

$$\boldsymbol{u_i} \cdot n_{/\Gamma_R} - T_n(\phi_{e/\Gamma_R}) \ = \ \boldsymbol{u_e} \cdot n_{/\Gamma_R} - \frac{\partial \phi_e}{\partial n}_{/\Gamma_R}. \tag{C.9}$$

From (C.5),(C.9) et (C.7), the problem to solve is written: find $(u_e, \phi_e) \in V_\beta(\Omega_e) \times H^1(\Omega_e)$ such that ($u_i$ is the data of the problem)

$$\begin{cases} \triangle_\beta \phi_e = 0 & \text{in } \Omega_e \\ \boldsymbol{u_e} \cdot n + T_n(\phi_e) - \dfrac{\partial \phi_e}{\partial n} \ = \ \boldsymbol{u_i} \cdot n & \text{on } \Gamma_R \\ \operatorname{rot}^\star_\beta(\operatorname{rot}_\beta u_e) = \dfrac{\omega^2}{c_\infty^2}[u_e - \nabla_\beta \phi_e] & \text{in } \Omega_e \\ (u_e \wedge n) \wedge n \ = \ (u_i \wedge n) \wedge n & \text{on } \Gamma_R. \end{cases} \tag{C.10}$$

Let us now admit, for a while, that the previous problem has a solution. The function $\phi_e$ being constructed, we recall that $\phi_i \in H^1(\Omega_i)$ is then the solution of (C.3 (i)) with the boundary condition $\phi_i = \phi_e$ on $\Gamma_R$ and we have constructed $u_e$ and $\phi_e$ in such a way that the conditions (C.3 (ii) and (iii) ) are satisfied. The last point to clarify, is to see if the problem (C.10) has a solution. This is the object of

◆

**Lemma C.2** *Let be $\psi \in H^{-\frac{1}{2}}(\Gamma_R)$, $\varphi_T \in H^{-\frac{1}{2}}(rot_\beta, \Gamma_R)$ and two strictly positive reals $\omega$ and $\beta$ such that*

$$\beta^2 - \frac{\omega^2}{c_\infty^2} > 0.$$

*The problem*

$$(Q_e) \quad \begin{cases} \text{Find } (u_e, \phi_e) \in V_\beta \times H^1(\Omega_e) \quad \text{such that} & \\ \triangle_\beta \phi_e = 0 & \text{in } \Omega_e \\ rot_\beta^\star(rot_\beta u_e) = \dfrac{\omega^2}{c_\infty^2}[u_e - \nabla_\beta \phi_e] & \text{in } \Omega_e \\ \boldsymbol{u_e} \cdot n + T_n(\phi_e) - \dfrac{\partial \phi_e}{\partial n} = \psi & \text{on } \Gamma_R \\ (u_e \wedge n) \wedge n = \varphi_T & \text{on } \Gamma_R \end{cases}$$

*has a unique solution*

**Proof** The proof is divided in two steps. The first one is to show the equivalence between the problem $(Q_e)$ and a variational problem. The second step consists in showing the existence and uniqueness of this last problem.

**(1)** - We can rewrite the problem $(Q_e)$ as an homogeneous one with the help of the field $z_e$, solution of $(P^e_{\omega,\beta})$ associated to $\varphi_T$. Indeed, $z_e$ satisfying

$$\begin{cases} \mathrm{rot}_\beta^\star(\mathrm{rot}_\beta z_e) = \dfrac{\omega^2}{c_\infty^2} z_e & \text{in } \Omega_e \\ (z_e \wedge n) \wedge n = \varphi_T & \text{on } \Gamma_R \end{cases}, \tag{C.11}$$

from the definition of $S_R(\omega, \beta)$, it verifies also the boundary condition $\boldsymbol{z_e} \cdot n = S_R(\omega, \beta)(\varphi_T)$, and setting $w_e = u_e - z_e$, $(Q_e)$ is the equivalent to

$$\begin{cases} \text{Find } (w_e, \phi_e) \in V_\beta^0(\Omega_e) \times H^1(\Omega_e) \quad \text{such that} & \\ (i) \quad \triangle_\beta \phi_e = 0 & \text{in } \Omega_e \\ (ii) \quad \mathrm{rot}_\beta^\star(\mathrm{rot}_\beta w_e) = \dfrac{\omega^2}{c_\infty^2}(w_e - \nabla_\beta \phi_e) & \text{in } \Omega_e \\ (iii) \quad \boldsymbol{w_e} \cdot n + T_n(\phi_e) - \dfrac{\partial \phi_e}{\partial n} = -S_R(\omega, \beta)(\varphi_T) + \psi & \text{on } \Gamma_R. \end{cases} \tag{C.12}$$

We can write a variational formulation associated to the problem (C.12), where the spaces of test functions are in $V_\beta^0(\Omega_e) \times H^1(\Omega_e)$. Indeed, at first, from (C.12 (i),(iii)), we have:

$$\forall \phi \in H^1(\Omega_e) \quad (\nabla_\beta \phi_e, \nabla_\beta \phi) + < w_e \cdot n + T_n(\phi_e) + S_R(\omega, \beta)(\varphi_T) - \psi, \phi >_{\Gamma_R} = 0.$$

From (C.12 (ii)), we deduce that

$$\forall w \in V_\beta^0(\Omega_e) \quad (\mathrm{rot}_\beta w_e, \mathrm{rot}_\beta w)_{0,\Omega_e} + \frac{\omega^2}{c_\infty^2}(\nabla_\beta \phi_e - w_e, w)_{0,\Omega_e} = 0.$$

Finally, if we note that $< \boldsymbol{w_e} \cdot n, \phi >_{\Gamma_R} = - \int_{\Omega_e} \nabla_\beta \phi \cdot w_e \, dx$ since $w_e$ satisfies the free divergence_$\beta$ condition, we see that $(w_e, \phi_e)$ is solution of

$$\left\{ \begin{array}{l} \forall (\phi, w) \in H^1(\Omega_e) \times V^0_\beta(\Omega_e) \\ a_e(\begin{pmatrix} \phi_e \\ w_e \end{pmatrix}, \begin{pmatrix} \phi \\ w \end{pmatrix}) = \dfrac{\omega^2}{c^2_\infty} < \psi - S_R(\omega, \beta) \varphi_T, \phi_{/\Gamma_R} > \end{array} \right. \tag{C.13}$$

where

$$\left\{ \begin{array}{lcl} a_e(\begin{pmatrix} \phi_e \\ w_e \end{pmatrix}, \begin{pmatrix} \phi \\ w \end{pmatrix}) & = & \dfrac{\omega^2}{c^2_\infty} [ \displaystyle\int_{\Omega_e} \nabla_\beta \phi_e \cdot \nabla_\beta \phi \, dx + < T_n \phi_e, \phi > \\ & + & \displaystyle\int_{\Omega_e} (\nabla_\beta \phi_e \cdot w - w_e \cdot \nabla_\beta \phi) \, dx \, ] \\ & + & \displaystyle\int_{\Omega_e} \mathrm{rot}_\beta w_e \cdot \mathrm{rot}_\beta w \, dx - \dfrac{\omega^2}{c^2_\infty} \int_{\Omega_e} w_e \cdot w \, dx. \end{array} \right.$$

Before showing that the problem (C.13) has a solution, the question is to know, if a solution of (C.13) is also solution of $(Q_e)$. And the answer is positive, if we show that a solution of (C.13) satisfies (C.13) replacing the test functions $w \in V^0_\beta(\Omega_e)$ by fields $\tilde{w} \in H_0(\mathrm{rot}_\beta, \Omega_e)$. Indeed, $\mathcal{D}(\Omega_e)^3$ being included into $H^1(\Omega_e) \times H_0(\mathrm{rot}_\beta, \Omega_e)$, it is easy to interpret the variational problem (C.13) in terms of the initial boundary value problem $(Q_e)$.

From the orthogonal decomposition (2.4) (with the inner product $( \; , \; )_{0,\Omega_e}$) of $H_0(\mathrm{rot}_\beta, \Omega_e)$: $H_0(rot_\beta, \Omega_e) = V^0_\beta(\Omega_e) \oplus H_\beta(\Omega_e)^\perp$, let us show that if $(\phi_e, w_e)$ is solution of (C.13), then

$$a_e(\begin{pmatrix} \phi_e \\ w_e \end{pmatrix}, \begin{pmatrix} 0 \\ v^\perp \end{pmatrix}) = 0 \qquad \forall v^\perp \in H_\beta(\Omega_e)^\perp.$$

If $v^\perp \in H^\perp_\beta(\Omega_e)$ then $v^\perp = \nabla_\beta \widetilde{\phi}$ with $\widetilde{\phi} \in H^1_0(\Omega_e)$ (cf. Theorem 1.2), which gives the following expression:

$$\begin{array}{lcl} a_e(\begin{pmatrix} \phi_e \\ w_e \end{pmatrix}, \begin{pmatrix} 0 \\ v^\perp \end{pmatrix}) & = & \dfrac{\omega^2}{c^2_\infty} [ \, (\nabla_\beta \phi_e, \nabla_\beta \widetilde{\phi})_{0,\Omega_e} - (w_e, \nabla_\beta \widetilde{\phi})_{0,\Omega_e} ] \\ & & + \, (\mathrm{rot}_\beta w_e, \mathrm{rot}_\beta (\nabla_\beta \widetilde{\phi}))_{0,\Omega_e}. \end{array}$$

The nullity of the second term comes from the property: $\mathrm{rot}_\beta(\nabla_\beta \widetilde{\phi}) = 0)$. The first term corresponds to:

$$\frac{\omega^2}{c^2_\infty} [(\nabla_\beta \phi_e, \nabla_\beta \widetilde{\phi})_{0,\Omega_e} - (w_e, \nabla_\beta \widetilde{\phi})_{0,\Omega_e}] = a_e(\begin{pmatrix} \phi_e \\ w_e \end{pmatrix}, \begin{pmatrix} \widetilde{\phi} \\ 0 \end{pmatrix}),$$

which is equal to zero, because $\tilde{\phi}_{/\Gamma_R} = 0$ and $(\phi_e, w_e)$ is solution of (C.13).

**(2)** - The second step of this proof is to show that the variational problem (C.13) has a unique solution, obviously in $H^1(\Omega_e) \times V^0_\beta(\Omega_e)$. This can be done using Lax-Milgram's theorem (cf. [7]). The only difficulty is to show the coercivity of $a_e$ in $H^1(\Omega_e) \times V^0_\beta(\Omega_e)$ (the other continuity properties of $a_e$ and of the right hand side of (C.13) are straightforward). But we have:

$$\begin{aligned} a_e(\begin{pmatrix} \phi \\ w \end{pmatrix}, \begin{pmatrix} \phi \\ w \end{pmatrix}) &= \frac{\omega^2}{c_\infty^2}[\ \|\nabla_\beta \phi\|^2_{0,\Omega_e} + < T_n\phi, \phi >_{/\Gamma_R}\ ] \\ &+ \|\mathrm{rot}_\beta w\|^2_{0,\Omega_e} - \frac{\omega^2}{c_\infty^2}\|w\|^2_{0,\Omega_e}. \end{aligned}$$

Then, the coercivity of $a_e$ arises from the two following properties:

- $< T_n\phi, \phi >_{/\Gamma_R} \ \geq\ 0$ because $T_n$ is positive.
- $\exists \alpha > 0\ ;\ \forall w \in V^0_\beta(\Omega_e) \quad \|\mathrm{rot}_\beta w\|^2_{0,\Omega_e} - \frac{\omega^2}{c_\infty^2}\|w\|^2_{0,\Omega_e} \ \geq\ \alpha \|w\|_{H(\mathrm{rot}_\beta,\Omega_e)}$
  This assertion comes from Lemma 2.3.

♦

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